30 YEARS OF SUPERGRAVITY

Paris, October 2006.

Conférence dédiée à Joel Scherk

30 years is a long time

Que reste-t-il de nos amours,
que reste-t-il de ces beaux jours...?
(What remains of our loves,
what remains of these beautiful days...?
song by Charles Trénet).

Thirty years of supergravity. . .

We celebrate this week 30 years of supergravity. It is fitting that this takes place in Paris, because many of the important discoveries of supergravity were made here. Long live supergravitý

long live our friends in Parı́s.

I have been asked to give a general introduction to local susy. For the experts I have tried to make this interesting by giving at the end some historical background of the concepts we use everyday. (**Please comment, add, or correct me: I am learning and studying these issues**).

For the nonexperts it all looks now simple and clean, but that is historical falsification: supergravity was discovered by hard labor in tortuous ways.

PATHS TO SUPERGRAVITY

- Rigid susy (Golfand, Likhtman, Akulov, Volkov, Wess, Zumino, Ferrara, Freedman, de Wit . . .) can be made local. Precedent: rigid isospin invariance led to Yang-Mills theory (1954).

- Loop corrections to Einstein gravity coupled to spin 2,1,0,1/2 gave non-renormalizable divergences. ('t Hooft, Veltman, Deser, van N). Coupling to QED and other magic combinations of fields did not improve matters (Grisaru, van N). Veltman and Salam suggested including spin 3/2 to me.

- Supersymmetrize gravity (?). Then the susy partner of the spin 2 graviton has either spin 3/2 or spin 5/2.

Some preliminary ideas

- For local $\epsilon^\alpha(x)$ $(\alpha = 1,4)$ one expects gauge field with δ ? $= \partial_\mu \epsilon^\alpha(x)$. So $\psi_\mu^\alpha(x)$, a real anticommuting vector-spinor field with spin 3/2.

- From $[\delta_Q(\epsilon_1(x)), \delta_Q(\epsilon_2(x))] = \delta_P(\epsilon_1(x), \epsilon_2(x))$ one gets local translations. Hence gravity with spin 2 enters: **supergravity**.

- Thus at linearized level one needs a spin (2,3/2) doublet of rigid $N = 1$ susy.

- From tree unitarity a unique free field action for spin 3/2

$$\mathcal{L}_{RS} = -\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\rho\sigma}\partial_\rho \psi_\sigma$$

$$\bar{\psi}_\mu = \psi_\mu^\dagger i\gamma^0 = \psi_\mu^T C$$

(Adding $\bar{J}^\mu \psi_\mu = \bar{\psi}_\mu J^\mu$, require that $\bar{J}^\mu (\text{Prop})_{\mu\nu} J^\nu$ has only simple poles with positive residues. This also fixes a possible mass term $-\frac{1}{2}M\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu$).

- This action has the local gauge invariance $\delta\psi_\mu = \partial_\mu \epsilon(x)$: gauge invariance follows from unitarity.

- We must now put $\mathcal{L}_{RS}$ in curved space, following Cartan and Weyl, and add the Einstein-Hilbert action.

Three ways to construct a gauge theory:

- the gauge action first

(Freedman, N, Ferrara 1976; Deser, Zumino 1976)

- the local gauge algebra first

(Yang and Mills in 1956, Freedman and N in 1976)

- the matter action first

(N, Lecture Notes in Physics **116** (1979)).

Historically, the gauge action came first

$$\delta\psi_\mu = \frac{1}{\kappa}\left(\partial_\mu\epsilon + \frac{1}{4}\omega_\mu{}^{mn}\gamma_{mn}\epsilon\right)$$

gravity   rigid susy   Weyl (1929)

LEAVE $\omega_\mu{}^{mn}$ UNSPECIFIED FOR A MOMENT

The gauge action for simple ($N = 1$) sugra

The choice of a gauge action in $3+1$ dimensions becomes now clear

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{RS}$$

$$\begin{aligned} \mathcal{L}_{EH} &= -\frac{1}{2\kappa^2}\, e\, e_m{}^\nu e_n{}^\mu R_{\mu\nu}{}^{mn}(\omega) \\ \mathcal{L}_{RS} &= -\frac{1}{2}\, e\, \bar{\psi}_\mu \gamma^{\mu\rho\sigma} D_\rho(\omega)\psi_\sigma \end{aligned}$$

with $e = \det e_\mu{}^m$ and

$$D_\rho\psi_\sigma = \partial_\rho\psi_\sigma + \frac{1}{4}\omega_\rho{}^{mn}\gamma_{mn}\psi_\sigma$$

$\gamma_{mn} = \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)$ with strength one, and $\gamma^{\mu\rho\sigma} = e_m{}^\mu e_r{}^\rho e_s{}^\sigma \gamma^{mrs}$ with γ^{mrs} antisymmetric in m, r, s, also with strength one. The Dirac matrices γ^m are constant. *For the time being we leave $\omega_\mu{}^{mn}$ unspecified.*

• **Step I:** To find the transformation rules of local susy we begin with $\delta\psi_\mu = \frac{1}{\kappa} D_\mu \epsilon$ in $\mathcal{L}_{RS}$. One finds after partial integration of $\delta\bar{\psi}_\mu = \frac{1}{\kappa} D_\mu \bar{\epsilon}$

$$
\begin{aligned}
\delta_\psi \mathcal{L}_{RS} = & -\frac{e}{16\kappa} R_{\rho\sigma}{}^{mn} \bar{\psi}_\mu \{\gamma^{\mu\rho\sigma}, \gamma_{mn}\}\epsilon \\
& + \partial_\mu \left[-\frac{e}{2\kappa} \bar{\epsilon} \gamma^{\mu\rho\sigma} D_\rho(\omega)\psi_\sigma \right] \\
& + \text{ terms with } D_\mu e_\nu^m
\end{aligned}
$$

(We discuss the boundary terms later).

• The first term reduces to $\bar{\psi}\gamma\epsilon$ in 4 dimensions, and yields an Einstein tensor $G_m^{\ \mu}(e,\omega)$ times $\bar{\epsilon}\gamma^m\psi_\mu$. The variation of the vielbeins (still unknown) in $\mathcal{L}_{EH}$ also yields an Einstein tensor times the unknown δe_μ^m. In this way one **derives** the vielbein transformation law

$$
\delta e_\mu{}^m = \kappa \bar{\epsilon} \gamma^m \psi_\mu
$$

Step 2: One is left with the following four variations in $\mathcal{L}_{EH}$

(i) the variation of ω in $\mathcal{L}_{EH}$ (yields also terms with $D_\mu e_\nu{}^m$ due to partial integration)

(ii) the variation of ω in $\mathcal{L}_{RS}$

(iii) the variation of the vielbeins in $\mathcal{L}_{RS}$

(iv) the terms with $D_\mu e_\nu{}^m$ which were obtained when we partially integrated $D_\mu \bar{\epsilon}$

To simplify the evaluation of (iii) and (iv), we use two other ways to write the action

$$\mathcal{L}_{EH} = -\frac{1}{8\kappa^2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{mnrs}R_{\mu\nu}{}^{mn}(\omega)e_\rho{}^r e_\sigma{}^s$$

$$\mathcal{L}_{RS} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma$$

There is only one vielbein left in $\mathcal{L}_{RS}$. It is straightforward to evaluate these four variations, and they factorize (!) (Townsend)

$$\delta(\text{remaining})\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\epsilon_{mnrs}$$
$$\frac{1}{2\kappa^2}(\delta_{\text{susy}}\omega_\mu{}^{mn}e_\nu{}^r + \frac{\kappa}{6}\bar{\epsilon}\gamma^{mnr}D_\mu\psi_\nu)$$
$$\times\,(D_\rho e_\sigma{}^s - \frac{\kappa^2}{4}\bar{\psi}_\rho\gamma^s\psi_\sigma)$$

Note that the second factor is the field equation of the spin connection $\omega_\mu{}^{mn}$. So there are two versions of supergravity:

(A) The second-order formalism according to which

$$D_{[\rho}e_{\sigma]}{}^s = \frac{\kappa^2}{4}\bar{\psi}_{[\rho}\gamma^s\psi_{\sigma]}.$$

One can solve this equation for ${\omega_\mu}^{mn}$ and finds then

$$
\begin{aligned}
{\omega_\mu}^{mn} = {\widehat{\omega}_\mu}^{mn} \equiv {\omega_\mu}^{mn}(e) + \\
\frac{\kappa^2}{4}(\bar{\psi}_\mu \gamma^m \psi^n - \bar{\psi}_\mu \gamma^n \psi^m + \bar{\psi}^m \gamma_\mu \psi^n)
\end{aligned}
$$

where ${\omega_\mu}^{mn}(e)$ is the usual textbook spin connection, a composite field depending on e_μ^m

$$
\begin{aligned}
\omega_{\mu mn}(e) = \left[\frac{1}{2} {e_m}^\nu (\partial_\mu e_{n\nu} - \partial_\nu e_{n\mu}) - m \leftrightarrow n\right] \\
- \frac{1}{2} {e_m}^\rho {e_n}^\sigma (\partial_\rho {e_\sigma}^c - \partial_\sigma {e_\rho}^c) e_{c\mu}
\end{aligned}
$$

and the $\bar{\psi}\gamma\psi$ terms are torsion. This is the solution of Freedman et al. in [1].

(B) The first-order formalism in which ${\omega_\mu}^{mn}$ is an independent field, whose variation is given by requiring the first factor to vanish. One can solve for $\delta_{\text{susy}}\omega_\mu{}^{mn}$ the same way as one solves for $\omega_\mu{}^{mn}$ and finds then

$$\delta_{\text{susy}}\omega_{\mu mn} = -\frac{1}{2}\bar{\epsilon}\gamma_5\gamma_\mu\tilde{\psi}_{mn} + \frac{1}{4}\bar{\epsilon}\gamma_5(\gamma^\lambda\tilde{\psi}_{\lambda n}e_{m\mu} - m \leftrightarrow n)$$

where $\tilde{\psi}_{mn} = \frac{1}{2}\epsilon_{mn}{}^{rs}\psi_{rs}$ and $\psi_{\mu\nu} = D_\mu(\omega)\psi_\nu - D_\nu(\omega)\psi_\mu$. This is the solution of Deser and Zumino in [2].

[C] One can combine the virtues of both formalisms into "1.5 order formalism"; use second-order formalism **but never vary** $\omega_\mu{}^{mn}(e,\psi)$ **in the action** because it is always multiplied with its own field equation which vanishes [3]. (see the factorized $\mathcal{L}$).

[1] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D **13** (1976) 3214;

[2] S. Deser and B. Zumino, Phys. Lett. B **62** (1976) 335;

[3] A. Chamseddine, P. Townsend, P. van Nieuwenhuizen, P. West.

- The 1.5 order formalism makes the tedious calculations of 4-fermions terms unnecessary. Very useful $D = 11$ sugra.
  2 order: E.Cremmer, B.Julia, J.Schark, Phys. Lett. **76** B (1978) 409
  1 order: L.Castellani, P.Fré, F.Giani, K.Pilch, P.van Nieuwenhuizen, Ann. Phys. **146** (1983) 35.
- The Einstein field equation for spin 3/2 "matter" reads in 1.5 order

$$\frac{e}{\kappa^2}(R_\nu{}^\tau - \frac{1}{2}\delta_\nu{}^\tau R) = \frac{i}{2}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma\epsilon^{\mu\tau\rho\sigma} \equiv \theta_\nu{}^\tau$$

  and the spin 3/2 field equation reads

$$R^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho(\omega)\psi_\sigma = 0.$$

  On-shell, $R = 0$. The consistency condition $D_\mu R^\mu = 0$ is satisfied (as first shown by Deser and Zumino in their first order approach).

- The spin 3/2 stress tensor T_m^{μ} is **not** θ_m^{μ} but

$$T_m{}^{\mu} = \theta_m{}^{\mu} - \frac{e}{\kappa^2}(G_m{}^{\mu}(e,\omega(e,\psi)) \\ - G_m{}^{\mu}(e,\omega(e))).$$

On-shell one finds

$$T_m^{\mu} = \frac{e}{\kappa^2} G_m^{\mu}(e,\omega(\mathbf{e}))$$

Clearly T_m^{μ} is conserved and symmetric on-shell.

- Finally: $\frac{e}{\kappa^2} G_\nu{}^\tau - \theta_\nu{}^\tau = 0$: all marble! hep-th/0606075

The local gauge algebra in x-space

The local gauge algebra [4] extends the rigid susy algebra. In particular, on $e_\mu{}^m$

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_E(\xi^\mu = \bar{\epsilon}_2 \gamma^\mu \epsilon_1)$$
$$+ \delta_{susy}(\epsilon = -\xi^\mu \psi_\mu) + \delta_{lL}(\lambda^{mn} = \xi^\mu \omega_\mu{}^{mn})$$

But on ψ_μ one find extra terms with the ψ_μ field equation. Auxiliary fields S, P, A_μ remove them [5]. Then

$$\mathcal{L}(\text{aux}) = -\frac{e}{3}(S^2 + P^2) + \frac{e}{3}(A_m)^2$$
$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \cdots + \delta_{lL}(\widehat{\lambda}^{mn})$$
$$\widehat{\lambda}^{mn} = \xi^\mu \widehat{\omega}_\mu{}^{mn} + \bar{\epsilon}_2 \gamma^{mn}(S - i\gamma_5 P)\epsilon_1$$

The concept of "supercovariant derivatives" and supercovariant tensors (Breitenlohner) simplifies formulas a lot. For example $\widehat{\omega}_\mu{}^{mn}$ is supercovariant: its susy variation contains no $\partial_\mu \epsilon$ terms

$$\delta\widehat{\omega}_\mu{}^{mn} = \frac{1}{4}\bar{\epsilon}(\gamma_b \hat{D}_\mu \psi_a - \gamma_a \hat{D}_\mu \psi_b - \gamma_\mu \hat{D}_a \psi_b)$$

The transformation rules of these auxiliary fields S, P, A_μ must be linear in the spin 3/2 field equation and they should be super covariant. They read

$$\begin{aligned}
\delta S &= \frac{1}{4}\bar{\epsilon}\gamma^\mu \widehat{R} \\
\delta P &= -\frac{i}{4}\bar{\epsilon}\gamma_5\gamma^\mu \widehat{R}_\mu \text{ (with } \gamma_5^2 = 1) \\
\delta A &= \frac{3i}{4}\bar{\epsilon}\gamma_5 \left(\widehat{R}_\mu - \frac{1}{3}\gamma_\mu\gamma^\nu \widehat{R}_\nu\right) \qquad (1)
\end{aligned}$$

The spin 3/2 transformation law contains auxiliary fields

$$\delta\psi_\mu = \frac{1}{\kappa}\left(D_\mu + \frac{i\kappa}{2}A_\mu\gamma_5\right)\epsilon - \frac{1}{2}\gamma_\mu\eta\epsilon \qquad (2)$$

where $\eta = -\frac{1}{3}(S - i\gamma_5 P - i\gamma^\mu A_\mu\gamma_5)$. This fixes $\widehat{R}_\mu$

$$\widehat{R}^\mu = \gamma^{\mu\nu\rho}\left(D_\nu\psi_\rho - \frac{i}{2}A_\rho\gamma_5\psi_\nu + \frac{1}{2}\gamma_\rho\eta\psi_\nu\right). \qquad (3)$$

[4] D. Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. D **14** (1976) 912.

[5] S. Ferrara, and P. van Nieuwenhuizen, Phys. Lett. **74** B (1978) 333;
K. Stelle and P. West, ibidem 330

In supergravity in superspace, the transformation rules before choosing a WZ gauge are linear in fields, and the gauge algebra closes in an obvious way. The minimal auxiliary fields in superspace were found in linear form by Ferrara, Zumino, and in nonlinear form by Siegel.

Superspace Supergravity

Superspace is due to Salam and Strathdee. Consider $\mathcal{L} = -\frac{1}{2}(\bar{\psi} + \bar{\lambda}\not{\partial})\not{\partial}(\psi + \not{\partial}\lambda)$. Local gauge invariance: $\delta\psi = -\not{\partial}\epsilon, \delta\lambda = \epsilon$. In unitary gauge $\lambda = 0$ one finds the Dirac action. But adding

$$\mathcal{L}_{\text{fix}} = \frac{1}{2}(\bar{\psi} - \bar{\lambda}\not{\partial})\not{\partial}(\psi - \not{\partial}\lambda)$$

one gets

$$\mathcal{L} = -\bar{\psi}\Box\lambda - \bar{\lambda}\Box\psi + \mathcal{L}_{\text{ghost}}.$$

With these ideas Yang Mills theory in superspace was constructed. For superspace supergravity one gets an extra vector index: $H_\mu(x,\theta)$. The action is now of the same generic form as super YM, namely $H_\mu DDD\cdots H_\nu$. Ogievetsky and Sokatchev

(Nucl. Phys. B **124** (1977) 309) proposed $H_\mu(x,\theta)\pi^{\mu\nu}H_\nu(x,\theta)$, where $\pi^{\mu\nu} = \eta^{\mu\nu}\Box +$ terms with D_α is a projector and where H_μ contains vielbein, gravitino, the auxiliary fields S, P, A_μ, and further auxiliary fields. Also Akulov, Soroka, Volkov (Th. Math. Phys. **31** (1977) 285), and Ferrara, Zumino (Nucl. Phys. B **134** (1978) 301) obtained similar linear results. The full nonlinear superspace formulation of supergravity was obtained by W. Siegel (1977 Harvard preprints; scans available on-line at KEK listed at SPIRES). Wess and Zumino developed a geometry of superspace with supervielbeins, and with constraints on supertorsions and supercurvatures. Siegel solved these constraints. See the talks by Wess, Zumino and Grisaru.

Conformal Supergravity

$N = 1$: Kaku, Townsend, N (1978-1980)
After many simplifications the complete action is NOW very simple.

$$\left.\begin{aligned}\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}(R_{\mu\nu}{}^{mn}(L)R_{\rho\sigma}{}^{rs}(L)\epsilon_{mnrs} \\ +\bar{R}_{\mu\nu}(Q)\gamma_5 R_{\rho\sigma}(S) \\ + R_{\mu\nu}(A)R_{\rho\sigma}(D))\end{aligned}\right\}$$

Here L, D, P, K, Q, S, D, A are generators

From "gauging" of $SU(2,2|1)$. Invariant under **all** (24) local symmetries if

$$R_{\mu\nu}{}^{m}(P) = 0 \quad \text{torsion, fixes } \omega_{\mu}{}^{mn}$$

$$\gamma^{\mu}R_{\mu\nu}(Q) = 0 \quad \text{fixes conformal gravitino}$$

$$R_{\mu\nu}{}^{mn}(M)e_n^{\nu}e_{m\rho} + R_{\mu\rho}(D) + \bar{\psi}^{\lambda}\gamma_{\lambda}R_{\rho\lambda}(Q) = 0$$

(fixes conformal vielbein)

Field eqs. of ordinary sugra become constraints of conformal sugra. These constraints are not field eqs. of conformal sugra.

The Weyl-gauge field for local scale transformations (1918) drops out of the action!

review: hep-th/0408137

By coupling the gauge fields of conformal $N=1$ supergravity to the fields A, B, χ, F, G of a WZ multiplet, and fixing the purely conformal symmetry by

$$A = 1 \quad (D); B = 0 \quad (A)\chi \quad \text{(linear comb. of } Q \text{ and } S) \qquad (4)$$

one is left with ordinary $N=1$ supergravity with fields $e_\mu{}^m, \psi_\mu, A_\mu, F, G$. From $|(\partial_\mu - ieA_\mu)A|^2$ one gets the auxiliary fields A_μ of ordinary sugra, and F, G become S, P.
Kaku and Townsend, Phys. Rev. **76** D (1978) 54
In superspace, the superfield $H_\mu(x,\theta)$ can also be used for conformal supergravity, depending on its coupling. The previous constraints on A, B, χ can then be written as a constraint on the corresponding chiral superfield (Siegel 1977).

$N = 2$ extended sugra

Unifies EM and gravity (Einstein's dream). Combine (2,3/2) with (3/2,1) to get extra $U(1)$

$$\mathcal{L} = -\frac{e}{2\kappa^2}R(e,\omega) - \frac{1}{2}\bar{\psi}_\mu i\gamma_5\gamma_\nu D_\rho(\omega)\psi_\sigma\epsilon^{\mu\nu\rho\sigma}$$
$$-\frac{1}{4}eF_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\varphi}_\mu i\gamma_5\gamma_\nu D_\rho(\omega)\varphi_\sigma\epsilon^{\mu\nu\rho\sigma}$$
$$+\frac{1}{2}(\mathcal{L} + \widehat{\mathcal{L}}_N)$$

where $\mathcal{L}_N$ couples ψ_μ to the spin (3/2,1) Noether current

$$\mathcal{L}_N = \frac{\kappa}{\sqrt{2}}\bar{\psi}_\mu(F^{\mu\nu} + i\gamma_5 F^{\mu\nu})\psi_\nu$$

There is a second local susy due to $\psi_\mu \to \varphi_\mu$ and $\varphi_\mu \to -\psi_\mu$ ($U(1)$ symmetry). Here ${\omega_\mu}^{mn}$ is supercovariant under both local susys. All 4-fermion terms are absorbed by 1.5 order formalism.

S.Ferrara, P.van N, PRL **37** (1976) 1669.

The formulations of $N = 1$ sugra in $d = 4$ are completely understood, both in x-space and in superspace. In time many extensions were created:

- extended sugras
- quantum corrections
- matter couplings
- higher and lower dimensions
- Kaluza-Klein reductions
- geometries

etc. etc.

However, one problem was left aside: the total derivatives due to partial integrations. Recently I have begun studying these. What follows is new material.

Boundary Conditions (BC) and Boundary Terms (BT)

$$\text{BC from} \quad \left.\begin{array}{l} \text{1) EL field eqs.} \\ \\ \text{2) Symmetries} \end{array}\right\} \text{not the same}$$

3) AND VARIATIONS THEREOF

BC on fields AND parameters (=ghosts in BRST). Consistency requires that the BC in 1) and 2) are invariant under 2):

"A theory with fields ϕ is invariant under a symmetry $\phi \to \tilde{\phi}$ if nothing changes when written in terms of $\tilde{\phi}$. So BC in ϕ must be the same as BC in $\tilde{\phi}$"

Lindström, Rocek, N; Nucl. Phys B **662** (2003)

This yields an "orbit of BC". One sometimes need to **add BT** to achieve consistency.

- In gravity, York, Gibbons and Hawking found

$$\delta S_{EH} + \delta S_{ext.\ curv.} = \int_M G^{\mu\nu} \delta g_{\mu\nu} + \int_{\partial M} K^{ij} \delta g_{ij}$$

GH imposed $\delta g_{ij} = 0$ at $\partial M = 0$
We found in sugra: $K^{ij} = 0$ at $\partial M = 0$
Vassilevich+N, Cl.Qu.Gr. **22** (2005) 5020.

One immediately finds for $N = 1$ sugra the following BT for local susy

$$\partial_\mu \left[-\frac{e}{\kappa^2} \delta \omega_\nu{}^{mn} e^\nu_m e^\mu_n - \frac{e}{2} \bar{\epsilon} \gamma^{\mu\rho\sigma} D_\rho(\omega) \psi_\sigma \right]$$

Comments:

• In superspace $\int d\theta^{\alpha} \to D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + \bar{\theta}^{\dot{\alpha}}\partial_{\dot{\alpha}\alpha}$

Boundary terms!

• One would like to impose only BC for supersymetries on off-shell fields. But consistency brings in some (?) of the BC for EL. M. Belyaev and N, in progress.

• BC should not depend on model:

$$\xi^{\mu} = \bar{\epsilon}_2\gamma^{\mu}\epsilon_1 : \text{ BC compatible.}$$

• Application: AdS/CFT program: Boundary action added but consistency of local susy of "bulk + boundary theory" not yet studied.

• Another interesting problem: Horava-Witten theory.

Ian Moss.

Conclusions:

30 years you and I have struggled with many collaborators to understand, apply and extend supergravity. Endless travels to exotic places, always to meet the same set of physicists, sometimes more heat than light, late nights in old buildings, despair when calculations turned out wrong, confusion that turned after hard work into more confusion: it has been a wonderful life. I thank all of you for making this possible. Let us hope that Nature is aware of our efforts.

A brief history of spinors

1) In math, E. Cartan (Bull. Soc. Math. France **41** (1913) 53) found spinors* as representations of orthogonal groups. ("Theorie des Spineurs" appeared in 1937 in Paris, and in 1966 at MIT Press). R. Brauer and H. Weyl ("Spinors in n dimensions" Ann. J. Math. **57** (1935) 425) expanded this work.

2) In physics, S. Goudsmit and G. Uhlenbeck (Nature **13** (1925) 953) introduced half-integer spin for electrons to explain the Zeeman effect. Compton (1923)

*Felix Klein even earler from conformal maps $S^2 \to z$ plane.

had already speculated on intrinsic angular momentum of photon.

3) Heisenberg and Jordan (ZfP **37** (1926) 263) gave **operator** formalism of spin with 2×2 matrices $\hat{s}_k$. So the "Pauli" matrices are due to the Heisenberg and Jordan.

4) Pauli (ZfP **43** (1927) 601) introduced 2-component spinors into nonrelativistic QM on which these $\hat{\sigma}_k$ act. "Zur Quantenmechanik des magnetischen Elektrons".

5) Fully relativistic theory of electrons with complex 4-component spinors by Dirac (1928).

6) In math Cartan (1922) introduced a geometry with VIELBEINS* and TORSION. ("On manifolds with an affine connection and the theory of general relativity", 1992 ENS; 1955 Gauthier Villars; 1986 Bibliopolis). Repères mobiles = vielbeins. On group (or coset) manifolds the relative orientations are fixed by group action (two nearby points = a vector).

*Earlier Darboux had already introduced Vielbeins, and Levi-Civita had already studied spin connections.

But Cartan also considered arbitrary frames. $A^\mu(x) = e_m^{\ \mu}(x) A^m(x)$: two arbitrary connections, $\Gamma_{\mu\nu}{}^\rho$ for A^μ and $\omega_\mu{}^m{}_n$ for A^m, related by compatibility = vielbein "postulate"

$$\partial_\mu e_\nu{}^m - \Gamma_{\mu\nu}{}^\rho e_\rho{}^m + \omega_\mu{}^m{}_n e_\nu{}^m = 0$$

$$e_\mu{}^m(x_P) = \partial\xi^m/\partial x^\mu \Big|_{x=x_P} .$$

The vielbein postulate has a geometrical interpretation: parallel transport commutes with conversion of flat indices to curved indices (or vice versa).

In other words parallel transport of A^μ and $A^m \equiv A^\mu e_\mu^m$ is the same for **any** $\omega_\mu{}^m{}_\nu$ and $\Gamma^\rho_{\mu\nu}$. Length is preserved if

$$\omega_\mu{}^{mn} = -\omega_\mu{}^{nm} \leftrightarrow D_\mu g_{\nu\rho} = 0$$

The $(e_\mu{}^m, \omega_\mu{}^{mn})$ formulation is completely equivalent to the $(g_{\mu\nu}, \Gamma_{\mu\nu}^\rho)$ formulation:

$$R_{\mu\nu}{}^{mn}(\omega) = R_{\mu\nu\rho}{}^\sigma(\Gamma) e^{\rho n} e_\sigma{}^m$$

where $R_{\mu\nu}{}^{mn}(\omega)$ is the YM curvature for the Lorentz group

$$R_{\mu\nu}{}^{mn}(\omega) = \partial_\mu \omega_\nu{}^{mn} + \omega_\mu{}^{ms} \omega_{\nu s}{}^n - (\mu \leftrightarrow \nu)$$
$$R_{\mu\nu\rho}{}^\sigma(\Gamma) = \partial_\mu \Gamma_{\nu\rho}{}^\sigma + \Gamma_{\mu\lambda}^\sigma \Gamma_{\nu\rho}^\lambda - (\mu \leftrightarrow \nu)$$

7) In physics, Einstein (1928) introduced "n-Beins": **fixed** vielbeins. His connection: $\Gamma_{\mu\nu}^{\rho} = -e^{\rho}{}_{m}\partial_{\mu}e^{m}_{\nu}$ was pure gauge ("Fernparallelismus".)

(He proposed $\Gamma_{\mu}{}^{\mu}{}_{\nu} - \left\{ {}_{\mu}{}^{\mu}{}_{\nu} \right\} =$ tensor $=$ EM field . Later he proposed to use torsion $\Gamma_{\mu}{}^{\mu}{}_{\nu} - \Gamma_{\nu}{}^{\mu}{}_{\mu} = A_{\mu}$.)

But Cartan had already introduced rigid and local vielbeins! In Review of 1930 by Einstein, Cartan added his views in an appendix ("not about priority").

8) Wigner (ZfP 1928) applied this to Dirac electron ψ (1928). Spinors in curved space need vielbeins. ψ is a coordinate scalar (!) and $k = 1, 2, 3$ is a "curved index". He symmetrized the operators

$$\frac{1}{2}\left[\gamma^n e_n{}^k (p_k + \epsilon A_k) + (p_k + \epsilon A_k)\gamma^n e_n{}^k\right]\psi = m\psi$$

Imposing coordinate invariance, he got

$$\left[\gamma^n e_n{}^k \left(\partial_k - \frac{i\epsilon}{\hbar}A_k\right) + \frac{1}{2}\gamma^n \frac{1}{e}\partial_k (e_n{}^k e)\right]\psi = m\psi$$

where $e = \det e_\mu{}^m$. (Correct but weird). He noted that $\frac{1}{e}\partial_k(e_n{}^k e)$ is a coordinate scalar! No local Lorentz invariance, no spin connection. But he noted rigid Lorentz invariance.

9) Weyl (ZfP 1929) discovers gauge theory* (refers to Wigner, Einstein, not Cartan). Main idea: the ${e_\mu}^m$ are ARBITRARY at each point. He put the Dirac action (1928) into curved space

$$\mathcal{L}_D = -e\bar{\psi}\left[\gamma^\mu\left(\partial_\mu - \frac{i\epsilon}{\hbar c}A_\mu + \frac{1}{4}{\omega_\mu}^{mn}\gamma_m\gamma_n\right) + \frac{mc}{\hbar}\right]\psi;$$

To obtain this result he proceeded as follows:

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x)$$

$$\gamma_\mu(x) = e_\mu^m(x)\gamma_m \qquad (I)$$

$$e_\mu^m e_\nu^n \eta_{mn} = g_{\mu\nu}(x) \qquad (II)$$

$$\sqrt{-\det g_{\mu\nu}} = \det e_\mu^m \equiv e$$

*But Hilbert had already noticed in 1916 that there is "overdetermination" and "underdetermination" in the Cauchy (initial value) problem for General Relativity).

(I) 16 equations for the 16 $e_\mu{}^m$

(II) 16-6=10 components $\Rightarrow$ new gauge symmetry: local Lorentz invariance:

$$\delta e_\mu^m = \lambda^m{}_n(x) e_\mu^n.$$

Then for ψ^α

$$\delta(\text{rigid})\psi = (-\lambda^\mu{}_\nu x^\nu)\partial_\mu\psi + \frac{1}{4}\lambda^{mn}\gamma_m\gamma_n\psi$$

but

$$\delta(\text{local})\psi = \frac{1}{4}\lambda^{mn}(\mathbf{x})\gamma_m\gamma_n\psi.$$

One needs a gauge field $\omega_\mu{}^{mn}$ for local Lorentz symmetry. Weyl defined it as $\omega_\mu{}^m{}_n = \left[e^{-1}(\partial_\mu + \Gamma_\mu)e\right]^m{}_n$. This is the vielbein postulate, but in reverse order. This spin connection $\omega_\mu{}^{mn}$ is a covariant Einstein vector, but a local Lorentz connection.

We know at this point that ${\omega_\mu}^{mn}$ transforms under spacetime gauge symmetries as follows

$$
\begin{aligned}
\delta_E {\omega_\mu}^{mn} &= \xi^\nu \partial_\nu {\omega_\mu}^{mn} + (\partial_\mu \xi^\nu) {\omega_\nu}^{mn} \\
\delta_{lL} {\omega_\mu}^{mn} &= -D_\mu \lambda^{mn} \\
&= -[\partial_\mu \lambda^{mn} + {\omega_\mu}^{ms} {\lambda_s}^n + {\omega_\mu}^{ns} \lambda^m].
\end{aligned}
$$

From its definition it is clear that the vielbein ${e_\mu}^m$ is also a covariant Einstein vector, and a local Lorentz vector.

$$
\begin{aligned}
\delta_E {e_\mu}^m &= \xi^\nu \partial_\nu {e_\mu}^m + (\partial_\mu \xi^\nu) {e_\nu}^m \\
\delta_{lL} {e_\mu}^m &= \lambda^m{}_n {e_\mu}^n
\end{aligned}
$$

Strategy for supergravity: covariantize at all stages w.r.t them; for example

$$
\delta \psi_\mu = \frac{1}{\kappa} D_\mu \epsilon = \frac{1}{\kappa} \left(\partial_\mu \epsilon + \frac{1}{4} {\omega_\mu}^{mn} \gamma_m \gamma_n \epsilon \right).
$$

10) The term “spinor” is due to Ehrenfest, who asked in a letter to van der Waerden, whether a “spinor analysis” existed (in flat space) similar to the “tensor calculus” (Pais, page 292). The answer contained the “dotted and undotted” indices: B.L. van der Waerden, Gött. Nachr. 100 (1929) and in “Die Gruppentheoretische Methode in der Quantum Mechanik, Springer, Berlin 1932”.

11) Higher spins:

M. Fierz and W. Pauli, Proc. Roy. Soc. A **173** (1939) 211.

M. Fierz, Helv. Phys. Acta **12** (1938) 3.

V. Bargmann and E. Wigner, Proc. Nat. Ac. Sci **34** (1948) 211: spinor wave functions, and repr. Poincare group.

12) To simplify "the clumsy formalism" (Schwinger's words) of Fierz and Pauli with multispinor indices and subsidiary constraints, Schwinger uses massive spin 3/2 fields ψ_μ^α to describe neutrinos in β decay

W. Rarita and J, Schwinger, Phys. Rev. **60** (1941) 61.

13) If one couples gravity with first-order ω to a Dirac electron, then from ω field equation one finds torsion

$$\tau_{\mu\nu}{}^{\rho} = \epsilon_{\mu\nu}{}^{\rho\sigma}\bar{\lambda}\gamma_5\gamma_\sigma\lambda$$

Then Dirac equation gets extra term

$$\gamma^\mu D_\mu\lambda + \gamma_5\gamma_m\lambda(\bar{\lambda}\gamma_5\gamma^m\lambda) = m\lambda = 0$$

(H. Weyl, Phys. Rev. **77** (1950) 699). (Hehl et.al., Rev. Mod. Phys. **48** (1976) 393.)

14) D.V. Volkov* (with Akulov and Soroka) GAUGED the super Poincaré algebra. Uses Cartan-Maurer equations for all

*hep-th/940453

connections. Uses a **NONLINEAR** realization of supersymmetry. Finds TORSION in rigid superspace! Introduces spin 3/2 fields gravitinos to obtain a super-Higgs effect. (But they do not obtain local susy, and find no $N \leq 8$ bound for supergravities).

15) 1976: The torsion introduced by Cartan in 1925 as a mathematical possibility, is finally realized in a concrete consistent quantum gauge theory by the spin 3/2 fields of sugra.

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The fantastic idea of a local gauge symmetry between bosons and fermions led to supergravity and to superstring theory. This should explain the basic structure of Nature

Darum hab ich mich der Magie ergeben, . . . dasz ich erkenne was die Welt im innersten zusammen hält. . .

Goethe, Faust

(Therefore I have surrendered myself to magic, . . . , so that I get to know what keeps the world together in its inner most reaches.)