Non-Renormalization Theorems

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Renormalizable QFT	Non-ren. QFT
$D=2$ σ-models	$D>2$ σ-model
	non-ren. potentials
$D=4$ WZ, SYM	$D>4$ SYM

flat superspace

Sugra — gauged → ungauged

curved superspace

Also,

pert. th. vs non-pert.

applies only when theory exists!
e.g. not to WZ model or non-ren. th.

Plan

- WZ model
- SYM
- ~~σ models & sugra~~
- sugra (1 transparency)

most of talk ∝ concentrate on pert. th.

WZ model : vacuum energy

$$I = \int d^4x\, d^4\theta\, \bar{\Phi}\Phi + \int d^4x \left[\int d^2\theta\, P(\phi) + \text{c.c.}\right]$$

$$P(\phi) = \frac{1}{2} m \phi^2 + \frac{1}{6} g \phi^3 \quad \text{(for renorm.)}$$

chiral superfield

complex constants

N.B. (i) no constant term in P : only P' appears in $\mathcal{L}_{cpt}$.
e.g $V \sim |P'|^2$

(ii) linear term in P can be removed by shift of ϕ when $m, g \neq 0$. $\Rightarrow \langle V \rangle_{vac} = 0$

As this follows from susy alone, it is also true in quantum theory

$\therefore$ no renorm. of vacuum energy

[Follows also from Witten index argument
Zeros of P' $\longrightarrow$ bosonic susy vacua

$$\therefore \quad \mathrm{tr}(-1)^F = \begin{cases} 2 & \text{if } g \neq 0 \\ 1 & \text{if } g = 0 \\ 0 & \text{if } m = g = 0, \text{ but } P \propto \phi \end{cases}$$
]

WZ model : renormalization

Loop calculations (Wess-Zumino, ...) indicated that one renorm. constant is sufficient!

$$\left.\begin{aligned} \phi &= Z^{1/2}\phi_R \\ g &= Z^{-3/2} g_R \\ m &= Z^{-1} m_R \end{aligned}\right\} \iff P \text{ unrenormalized}$$

Proved by
- Ward Identities — Iliopoulos & Zumino
- Superfield pert. th. — Grisaru, Roček & Siegal
- Holomorphy — Seiberg

N.B. Mass renorm. is multiplicative $\Rightarrow$ potential solution to heirarchy problem

If susy unbroken at $E > \Lambda_{susy}$ then contributions to ΔM^2_{Higgs} come only from $E < \Lambda_{susy}$

$\therefore$ Expect $m^2_{Higgs} \sim \Lambda_{susy}$ & sparticles at LHC!

Still need to explain why $\Lambda_{susy} \ll \Lambda_{Planck}$
but $\exists$ plausible scenarios involving e.g.

i) dyn. sym. breaking
ii) susy broken in hidden sector

1PI vs Wilson

Given massless particle, could get contribution to effective action of form

$$(\ ) \int d^4x \int d^4\theta \, \frac{1}{\Box} \left[\bar{D}^2 \Delta P(\phi) + c.c. \right]$$

finite

non-local in superspace

$$\sim \int d^4x \left[d^2\theta \, \Delta P(\phi) + c.c \right]$$

$\Rightarrow$ finite renormalization of P (Grisaru, Gates, Roček, Siegel)

Actually happens in massless $\begin{cases} \sigma\text{-models (Howe \& West)} \\ \text{WZ-model (Jack, Jones \& West)} \end{cases}$

This is clearly an <u>IR effect</u>

Can avoid this (and other) IR effects by considering <u>Wilsonian effective action</u>

i.e. Integrate out all modes with $E > M$ to get effective action with M-dep couplings.

N.B. (i) Wilson = 1PI if no massless particles

(ii) How would one carry out Wilson's computation preserving Lorentz inv. ?

Non-renorm. th. via prepotential

$$\int d^2\theta\, P(\phi) = \int d^2\theta\, f(\phi)\,\phi$$

complex unconstrained prepotential

$$= \int d^2\theta\, f(\phi)\,\bar{D}^2 Z \qquad \left(\text{since } \bar{D}\phi = 0 \Rightarrow \phi = \bar{D}^2 Z\right.$$

$$= \int d^4\theta\, f(\phi)\, Z$$

e.g. $P = \phi^3$: $\int d^2\theta\, P = \int d^4\theta\, \phi^2 Z$ (cf CS term)

Using background field method : $Z = Z_{cl} + \zeta$

$$\int d^2\theta\, P(\phi) = \underbrace{\int d^2\theta\, P(\phi_{cl})}_{} + \underbrace{\int d^4\theta\, F(\phi_{cl}, \zeta)}_{}$$

$$\int d^2\theta\, P(\phi_{cl}) = \underbrace{\int d^4\theta\, f(\phi_{cl})\, Z_{cl}}_{}$$

not manifestly pre-gauge inv.
∴ cannot appear as c.t.

∴ counterterms are manifestly pre-gauge inv. full superspace integrals

N.B. pregauge inv. $Z \to Z + \bar{D}_{\dot\alpha}\bar{\Lambda}^{\dot\alpha}$ requires ghosts & ghosts for ghosts ad infinitum

∴ argument fails at 1 loop. Need separate 1 loop argument (or calculation)

Holomorphy (Seiberg)

Wilsonian eff. action must have form

$$Z^{-1}\int d^4x\, d^4\Theta\, \bar{\Phi}\phi \;+\; \int d^4x \left[d^2\Theta\, P_{eff}(\phi, m, g) + c.c \right]$$

Know P_{eff} is holomorphic in ϕ

Assume " " " in m, g too

Note "invariances":

$U(1)$ invariants

$$U(1)\begin{Bmatrix} \phi \to e^{-i\alpha}\phi \\ m \to e^{2i\alpha} m \\ g \to e^{3i\alpha} g \end{Bmatrix} \Rightarrow P_{eff} = f(m\phi^2, g\phi^3)$$

holomorphic fn.

$$U(1)_R\begin{Bmatrix} \phi \to e^{-i\beta}\phi \\ m \to m \\ g \to e^{i\beta} g \\ \text{and } \Theta \to e^{i\beta}\Theta \end{Bmatrix} \Rightarrow f = \underbrace{\tfrac{1}{2}m\phi^2}_{\text{R weight 2}}\, h\left(\frac{g\phi^3}{3m\phi^2}\right)$$

R weight 2 to cancel R-weight of $d^2\Theta$ measure

R-inv. ratio

$h(\rho) = \sum_n h_n \rho^n$ but nth term $\sim g^n\phi^{n+2}$ $\Rightarrow$ tree approx

But $h = 1 + \rho$ in tree approx $\Rightarrow$ $P_{eff} = P$

[N.B. $U(1)_R$ is chiral - could be anomalous - but not for WZ]

Spurion justification

Consider m, g as $\theta = 0$ cpts of chiral superfields

$M = m + \cdots, \quad G = g + \cdots$ i.e. "spurions" (Cirandello & Grisaru

- Explains why PeH must be holomorphic in m, g
- Elevates "symmetries" that change m, g to genuine symmetries in which only fields transform.

Q. Can this be made rigorous?

Consider the $U(1) \times U(1)_R$ inv. action

$$I[\phi, M, G] = \int d^4x \int d^4\theta\, \bar{\Phi}\phi + \int d^4x \left\{ d^2\theta \left[\frac{1}{2} M\phi^2 + \frac{1}{6} G\phi^3 \right] + c.c. \right\}$$

$$+ \int d^4x\, d^4\theta \left[K_M \bar{M} M + \mu^2 K_G \bar{G} G \right]$$

Problems (i) $\exists$ non-renorm. interactions now

(ii) M G in loops

(iii) $M = m$, $G = g$ is inconsistent truncation

Plausibly, these problems go away as $K_M, K_G \to \infty$

but is there a proof?

$\mathcal{N}=4$ SYM : Early ideas

Loop calculations ($\ell = 1,2,3$) suggest $\beta \equiv 0$.

If so, why?

- Anomaly argument (Ferrara & Zumino, Sohnius & West)

(i) $\partial \cdot J_5$, $\gamma \cdot S$, $T_\mu{}^\mu$ in one chiral multiplet,

so $\partial \cdot J_5 = 0 \Rightarrow T_\mu{}^\mu = 0$

(ii) For $N=4$ viewed as $N=1$, $U(1) \subset SU(4)$
$\Rightarrow \partial \cdot J_5 = 0$ if $SU(4)$ preserved

But this requires prior resolution of the "anomaly puzzle"

$\partial \cdot J_5 \neq 0$ at 1 loop only (Adler-Bardeen)

$T_\mu{}^\mu \neq 0$ at 1 & 2 loops at least for generic $N=1$ theory (by computation)

How is this consistent with susy? (More later)

- $\left.\begin{array}{l}N=1 \text{ non-ren. th.} \\ SO(4) \text{ sym}\end{array}\right\} \Rightarrow$ no renormalization at all! (Stelle)

This is an improvement, but now we must use $N=1$ superfields so $SO(4)$ is not manifest

No go for $N=4$

(Rivelles & Taylor; Howe, Stelle, PKT)

Off-shell N-extended super-Maxwell has cpt Lag.

$$L = \Phi \, \mathcal{O} \, \Phi = L_{bos} + \underbrace{L_{fer}}_{\bar\lambda \not\partial \lambda + \bar\xi \psi}$$

all fields (Φ), diff. op. ($\mathcal{O}$)

N photini (λ), n aux. fermion pairs (ξ, ψ)

Off-shell susy implies

$$L' = \Phi \, \mathcal{O} \left(1 + \frac{\Box}{m^2}\right) \Phi \quad \underline{\text{also}} \text{ off-shell susy}$$

$$m^2 L'_{fer} = \underbrace{\bar\lambda \not\partial (\Box + m^2) \lambda}_{} + \underbrace{\bar\xi (\Box + m^2) \psi}_{}$$

on-shell state content: $2N$ massless, $4N$ massive ghosts; $4n$ massive, $4n$ massive ghosts

$$\therefore \left.\begin{aligned} 4n &= 2^{2N-1} p \\ 4N + 4n &= 2^{2N-1} q \end{aligned}\right\} \quad p, q \in \mathbb{Z}$$

$$\therefore \quad p - q = \frac{4N}{2^{2N-1}} \qquad (\text{assuming } n < \infty)$$

$\Rightarrow \underline{N = 1, 2}$

Q. Is $N=2$ possible / sufficient?

$\underline{N=2 \text{ pert. th.}}$ (Howe, Stelle, PKT)

$$I^{N=2}_{\text{vector}} = \int d^4x \left[\int d^4\theta \, \mathrm{tr}\, W^2 + c.c \right]$$

chiral $N=2$ complex scalar s.f.

chiral $N=2$ superspace

$$I^{N=2}_{\text{hyper}} = \int d^4x \, d^8\theta \left[L_{ij} D^{\dot\alpha}_{\alpha} \rho^{\alpha ij} + c.c. + L^{ijkl} X_{ijkl} \right]$$

+ covariantization

$N=2$ pre-potentials

on-shell fields are
$(S = D^{ijkl} X_{ijkl}, \ L_{ij})$

Details of background field quantization imply

> All counterterms are integrals of <u>local</u>, <u>manifestly</u> gauge & pre-gauge inv. functions of background fields, <u>except at 1-loop</u>

But $I^{N=2}_{\text{vector}}$ and $I^{N=2}_{\text{hyper}}$ do <u>not satisfy</u> this condition

- One loop exception because of no ghosts for ghosts
- Expect horrible IR divergence problems in practice
- <u>Only applicable for hypers in real reps of gauge group</u>

Finite $N=2$ theories

For hypermultiplet in (possibly reducible) rep R

$$\beta\Big|_{1\,\text{loop}} = \frac{g^3}{8\pi^2}\Big[\underbrace{T(R)}_{\text{quadratic casimir in rep } R} - \underbrace{C_2}_{\text{quadratic casimir}}\Big]$$

For $R =$ adjoint, $T(R) = C_2$ so $\beta\Big|_{1\,\text{loop}} = 0$

$\therefore$ finite to all order. This is the $N=4$ theory.

However, argument applies to any $N=2$ theory with $\beta = 0$ at 1 loop (Howe, Stelle, West)

Provided $l > 1$ result can be extended to any rep. R.

- Fortunately, $\exists$ much improved $N=2$ non-renorm. thm using $N=2$ harmonic superspace of Galperin et al.

This applies to any rep R, and is simpler (Buchbinder, Kuzenko, Ivanov & Ovrut)

- $N=4$ also proved UV finite by light-cone superspace methods (Mandelstam; Brink, ~~Nilsson~~ Lindgren & Nilsson)

<u>$D > 4$ max. SYM</u> (Howe & Stelle)

$$I_{YM} = \int d^Dx \, \frac{1}{g^2} \, \mathrm{tr}\, F^2 \qquad \text{mass. dim. of } g = -\frac{(D-4)}{2}$$

$\therefore$ more loops $\Rightarrow$ more derivs.

e.g. <u>$D=5$</u>. Expect 4-loop c.t. $\sim g^6 F^4$

Given <u>8 manifest susys</u> (½ max) this could appear as

$$g^6 \int d^5x \int d^8\theta \, W^4$$

complex scalar of max susy $D=5$ SYM mult.

Can't write $l < 4$ c.t. as ½ superspace integrals of cov. fields

$\Rightarrow$ <u>Expect 1st U.V. div. at $l = 4$</u>

However, there are indications (Bern et. al) that 1st U.V. div. occurs at $l = 6$

Explained by ¾-superspace parl. th., (Howe & Stelle)

$D=4$ $N=3$ is possible using harmonic superfields (as aux. fields so escapes no-go th'm) (~~Zupnik~~ Galperin, Ivanov, Kalitzin, Ogievetsky, & Sokatchev)

$D=5$ version would be non-Lorentz cov.!

Holomorphy for $N=1$ (Seiberg)

$$I = \int d^4x\, d^4\theta \left(\phi^+ e^{-V} \phi\right) + \underbrace{\int d^4x \left[\int d^2\theta \, \frac{1}{g_h^2} \operatorname{tr}(W^\alpha W_\alpha) + c.c\right]}_{\text{SYM}}$$

$$+ \int d^4x \left[d^2\theta\, P(\phi) + c.c.\right]$$

"holomorphic" coupling const

$$\frac{1}{g_h^2} = \frac{1}{g^2} + \frac{i\theta}{8\pi^2}$$

Dependence on g_h of effective (Wilsonian) kinetic term for vector fields is

(i) holomorphic : $\displaystyle \frac{8\pi^2}{g_h'^2} = \frac{8\pi^2}{g_h^2} + f\left(\frac{8\pi^2}{g_h^2},\ t = \log M'/M\right)$

coupling at scale M' — holomorphic — coupling at scale M

(ii) periodic : $\displaystyle f\left(\frac{8\pi^2}{g_h^2} + 2\pi i, t\right) = f\left(\frac{8\pi^2}{g_h^2}, t\right)$

$$\Rightarrow \beta\left(\frac{8\pi^2}{g_h^2}\right) = \sum_{n=0}^{\infty} a_n \left(e^{-8\pi^2/g_h^2}\right)^n$$

$$= a_0 + \text{non-pert. terms}$$

1 loop result

[N.B. $\beta(g) = \frac{d}{dt} g = -g^3 \frac{d}{dt}\left(\frac{1}{g^2}\right) \propto -g^3$ ✓]

$\therefore$ 1-loop β-fn. exact (cf. Adler-Bardeen)

Anomaly puzzle resolution + confusion

- Grisaru x Zanon : $J_5^\mu(\text{susy})$ and $J_5^\mu(AB)$ differ at quantum level.

[Comment: just as well! But have we really learnt something?]

- Shifman x Vainshtein : Contributions to $\beta(g)$ beyond 1 loop are IR effects

[Comment: So what are we supposed to do with IR effects — ignore them]

- Arkani-Hamed x Murayama : $\beta(g_h)$ exact at 1-loop but standard calcs in scheme with $F = dA + \frac{1}{2}g[A,A]$ are non-holomorphic.

Need field redefinition to go from one to other. Jacobian non-trivial

$$\beta(g_h) \longrightarrow \beta_{NSVZ} \quad (\text{all loop order})$$

one-loop exact

given 'exactly' in terms of anomalous dim.

[Comment: Sounds good!]

- Stepanyantz : $\beta(g)$ exact at 1-loop in higher-deriv. regularization

[Comment: True that this reg. is only one known that preserves susy, so sounds good too!]

Supergravity

- $N=8$ sugra has 7-loop counterterm $\int d^4x\, d^{32}\Theta\, W^8$ (Howe, Kholodsha)
  so $N=8$ superfield pert. th. would $\Rightarrow$ 1st U.V div. at 7 loops
  (Grisaru & Siegel).

  However we don't expect $\exists$ $N=8$ pert. th.
  If only $N=4$ is possible then $\exists$ 3-loop c.t.
  so expect U.V. div. at 3-loops

  Bern et al. suggest 1st U.V div. at $\ell=5$
  Could be explained by $N=6$ pert. th.

- $D=11$ sugra. Automatically finite at $\ell=1$
  At $\ell=2$ have $\partial^{12}R^4$ c.t.
  as full superspace integral (Deser)
  so no non-ren. th'm for $D=11$

- Conformal sugra $N=4$ is max. and application of non-renorm. th'm suggests that it is finite (Howe, Stelle, PKT)

[However we don't actually have $N\geqslant 4$ superfield pert. th. for sugras.]