Branes in Supergravity

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30 Years of Supergravity Conference

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Achucarro, Bergshoeff, Cremmer, Cvetic, Dabholkar, de Alwis, Duff, Gauntlett, Gibbons, Güven, Harvey, Howe, Hull, Kallosh, Liu, Lü, Lukas, Olive, Ovrut, Papadopoulos, Polchinski, Pope, Sezgin, Stelle, Strominger, Townsend, Tseytlin, Waldram, Witten *et al*.

Supersymmetric Solitons - some early milestones

- 1978: Witten & Olive - topological central charges in supersymmetry algebras
- 1982 onwards: study of Bogomol'ny bounds for solitons in supersymmetric theories, particularly by Gibbons, Manton, Osborn *et al.* in Cambridge
- 1984: Green & Schwarz - covariant superstring action
- 1986: Hughes, Liu & Polchinski - Green-Schwarz action for super 3-brane in D=6
- 1987: Bergshoeff, Sezgin & Townsend - Green-Schwarz worldvolume action for supermembrane in D=11 supergravity
- 1990: Strominger - Heterotic string 5-brane obtained by lifting YM instanton to D=10; Dabholkar, Harvey, Gibbons & Ruiz Ruiz - superstring solution to D=10 type 2A supergravity; Duff & Stelle - supermembrane solution to D=11 supergravity

Linguistic history

Oxford English Dictionary entry on first use of "brane":
in "Semiclassical quantization of the supermembrane,"
Duff, Howe, Pope, Sezgin & Stelle, Nucl. Phys. B297: 515, 1988.

(See alsoTownsend and Stelle, "Are 2-branes better than one?"
proceedings of the CAP Summer Institute, Edmonton, Alberta, July 1987)

The notation p for the spatial dimension of a generalized membrane is due to Paul Townsend.

Reissner-Nordström/Majumdar-Papapetrou black holes

- Ur-example of what we call a BPS brane: Majumdar-Papapetrou solution to D=4 Einstein-Maxwell theory

$$ds*2 = -H^{-2}dt^2 + H^2 dx^i dx^i$$
$$F = dt \wedge d\left(\frac{1}{H}\right)$$

where H is an arbitrary harmonic function on $\mathbb{E}^3$

- Solution as expressed is purely electric, but can be duality rotated to give a general dyonic black hole with (electric, magnetic) charges (q,p)

$$q = \frac{1}{4\pi}\int_{S^2_\infty} *F \qquad p = \frac{1}{4\pi}\int_{S^2_\infty} F$$

Bogomol'ny bound

- From Witten-Nester energy positivity arguments, obtain a bound $M \geq |Z|$ for the ADM mass of the RN/MP solution where $|Z|^2 = \frac{q^2 + p^2}{G}$
- Equality holds when there exists a Killing spinor field $\nabla_a \varepsilon = 0$, $\delta_\varepsilon(\text{spinor matter}) = 0$, so these extremal (BPS) solutions are supersymmetric.
- The harmonic function *H* can be chosen to have multiple singularities, corresponding to multiple-black-hole configurations with arbitrary locations for the holes. The BPS extremality condition translates to a *vanishing force* between similarly charged black holes: attractive gravitational and repulsive Maxwell forces exactly cancel in these static configurations.

Charges, Singularities and topology

- The RN/MP solutions display the essential features of the whole BPS family of supersymmetric brane solutions.
- A characteristic feature is the presence of electric or magnetic charges, corresponding to central charges in the supersymmetry algebra realizations for these solutions.
- Note, however, that the fundamental fields of the supergravity theory do not themselves carry such charges. So perturbative states cannot carry them, and the brane solutions are accordingly inherently non-perturbative. Such solutions can possess non-vanishing charges only if they are singular or topologically non-trivial or both.

The M_2 and M_5 branes in D=11 supergravity

Duff & Stelle
Güven

- Look for solutions analogous to the RN/MP solution in D=11 supergravity, whose bosonic sector has the action

$$I_{11} = \int d^{11}x \left\{ \sqrt{-g}\left(R - \tfrac{1}{48}G_{[4]}^2\right) + \tfrac{1}{6}G_{[4]} \wedge G_{[4]} \wedge C_{[3]} \right\}$$

where $C_{[3]}$ is a 3-form gauge field and $G_{[4]}=dC_{[3]}$ is its field strength.

- The equation of motion for the $C_{[3]}$ gauge field

$$\mathrm{d}\,{}^*G_{[4]} + \tfrac{1}{2}G_{[4]} \wedge G_{[4]} = 0$$

and the Bianchi identity $dG_{[4]} = 0$ give rise to "electric" and "magnetic" conserved charges

$$U = \int_{\partial\mathcal{M}_8} \left({}^*G_{[4]} + \tfrac{1}{2}C_{[3]} \wedge G_{[4]}\right)$$

$$V = \int_{\partial\mathcal{M}_5} G_{[4]}$$

$\mathcal{M}_8$, $\mathcal{M}_5$: 8 and 5 dim spacelike hypersurfaces; will be transverse spaces to brane worldvolumes

- The choices of space orthogonal to M_8 or M_5 give these charges natural 2-form or 5-form structures: $U_{[2]}$, $V_{[5]}$

- These charges appear in an extension of the D=11 supersymmetry algebra: $\{Q, Q\} = C(\Gamma^A P_A + \frac{1}{2}\Gamma^{AB} U_{AB} + \frac{1}{5!}\Gamma^{ABCDE} V_{ABCDE})$

  M-brane algebra *528=11+55+462* van Holten & van Proeyen

- They support electric 2-brane and magnetic 5-brane solutions

$$ds^2 = H^{-\frac{\tilde{d}}{9}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{d}{9}} dy^m dy^m \qquad d = \begin{cases} 3 & \text{2-brane} \\ 6 & \text{5-brane} \end{cases}$$

$$H(y) = 1 + \frac{k}{r^{\tilde{d}}} \qquad \mu = 0, 1, \ldots, p = d-1 \qquad \tilde{d} = \begin{cases} 6 & \text{2-brane} \\ 3 & \text{5-brane} \end{cases}$$

worldvolume

$$G_{m\mu_1\ldots\mu_3} = \varepsilon_{\mu_1\ldots\mu_3}\partial_m(H^{-1}) \qquad m = 3, \ldots, 10 \quad \text{2-brane}$$
$$G_{m_1\ldots m_4} = -\varepsilon_{m_1\ldots m_4 r}\partial_r H \qquad m = 6, \ldots, 10 \quad \text{5-brane}$$

transverse

- These can be generalized to allow *H(y)* to become an arbitrary harmonic function in the 11-*d* space transverse to the *d*-dim. worldvolume.

Bogomol'ny bounds

- Static branes of infinite extent have divergent ADM energy, but the energy per unit p-volume is finite and is given precisely by the corresponding charge:

$$\mathcal{E} = U = \int_{\partial M_8} d\Sigma_7^m G_{m012} = \tilde{d}k\Omega_7 \qquad \textit{energy/2-volume}$$

$$\mathcal{E} = V = \frac{1}{4!}\int_{\partial M_5} d\Sigma_4^m \epsilon_{mnpqr} G^{npqr} = \tilde{d}k\Omega_4 \qquad \textit{energy/5-volume}$$

- These relations saturate the corresponding Bogomol'ny bounds

$$\begin{aligned} \mathcal{E} &\geq U \\ \mathcal{E} &\geq V \end{aligned}$$

and accordingly preserve 1/2 supersymmetry, *i.e.* 16 out of 32 supercharges.

pp waves and NUTs

- In addition to the 2-brane and 5-brane, M-theory (i.e. D=11 supergravity) possesses two other purely gravitational brane types:

pp wave

$$ds_{11}^2 = \{-dt^2 + d\rho^2 + (H(y) - 1)(dt + d\rho)^2\} + dy^m dy^m$$
$$C_{[3]} = 0 \,, \qquad m = 2, \ldots, 10$$

Brinkman 1923

NUT

$$ds_{11}^2 = -dt^2 + dx_1^2 + \ldots + dx_6^2 + ds_{TN}^2(y)$$
$$C_{[3]} = 0$$
$$ds_{TN}^2 = H(y) dy^i dy^i + H^{-1}(y)(d\psi + V_i(y) dy^i)^2 \,, \qquad i = 1, 2, 3$$
$$\vec{\nabla} \times \vec{V} = \vec{\nabla} H$$

- These are also considered branes because they preserve 16 supercharges and because they generate important brane families under dimensional reduction. *E.g.* under reduction to D=10 the pp wave gives the IIA theory extremal black hole supported by the Kaluza-Klein vector, while the NUT gives the IIA 6-brane.

Quantization conditions and brane lattices

- Electric and magnetic brane types supported by the same field strength must obey a Dirac quantization condition generalizing the D=4 Maxwell-theory condition $qp = 2\pi n$: Nepomechie; Teitelboim/Bunster, ...

$$Q^{\rm el}_{[p]} \wedge Q^{\rm mag}_{[\hat{p}]} = 2\pi n \frac{Q^{\rm el}_{[p]} \wedge Q^{\rm mag}_{[\hat{p}]}}{|Q^{\rm el}_{[p]}|\,|Q^{\rm mag}_{[\hat{p}]}|}\,, \quad n \in \mathbb{Z}$$

- These need to be taken together with the D=11 restriction on the charge-lattic basis needed to ensure invariance under large 3-form gauge transformations

$$Q_5 = \frac{1}{2\pi} Q_2^2$$

de Alwis;
Lavrinenko, Lü, Pope & Stelle

- As a result, the *charge lattice* for *p*-branes in all dimensions $D \leq 11$ is entirely fixed.

Dimensional reduction brane families

Lü, Pope, Sezgin & Stelle

- Solving for the simplest type of p-brane after the various worldvolume or transverse dimensional reductions that can be made boils down to solving the field equations for a D-dimensional system containing the metric, a scalar ϕ and an n-form antisymmetric tensor field strength $F_{[n]}$: $I = \int D^D x \sqrt{-g}\left[R - \frac{1}{2}\nabla_M \phi \nabla^M \phi - \frac{1}{2n!} e^{a\phi} F_{[n]}^2\right]$
- Under further dimensional reduction, the descendants of such a system have different values of the coupling parameter a, but if one defines $a^2 = \Delta - \frac{2d\tilde{d}}{(D-2)}$; $\tilde{d} = D - d - 2$ then Δ is preserved for branes related by dimensional reduction.
- The parameter Δ governs the warp factors in the corresponding brane metric: $ds^2 = H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^m dy^m$

Intersecting branes

Papadopoulos & Townsend; Tseytlin, ...

- Solving for the various possibile cases of the simplified (metric, form-field & scalar) system, one realizes that in all cases $\Delta = 4/N,\ N \in \mathbb{Z}$, suggesting that in addition to the four basic brane "elements" there exists a richer "chemistry" of composite branes.
- This was realized with the finding of "intersecting branes" possessing reduced amounts of preserved supersymmetry.
- *Eg.* there is a 1/4 supersymmetric solution consisting of a 2-brane and a 5-brane intersecting over a 1-brane subspace,

$2 \perp 5(1)$

$$
\begin{aligned}
ds^2 \;=\; & H_1^{\frac{1}{3}}(y) H_2^{\frac{2}{3}}(y)[H_1^{-1}(y) H_2^{-1}(y)(-dt^2 + dx_1^2) \\
& + H_1^{-1}(y)(dx_2^2) + H_2^{-1}(y)(dx_3^2 + \ldots + dx_6^2) \\
& + dy^m dy^m] \qquad m = 7, \ldots, 10 \\
G_{m012} = \partial_m(H_1^{-1}) & \qquad G_{2mnp} = -\epsilon_{mnpq}\partial_q H_2
\end{aligned}
$$

$H_1(y)$, $H_2(y)$: independent harmonic functions

Neugebauer & Kramer;
Breitenlohner, Maison &Gibbons;
Gal'tsov & Rytchkov

Harmonic maps

- Above, we discussed the way in which the *p*-brane spectrum of supergravity can be derived from 4 "elements" in D=11 (M_2, M_5, wave & NUT).
  - A complementary way to understand the family of brane solutions is to dimensionally reduce on *all* worldvolume coordinates so that only the transverse dimensions remain. In this reduced (Euclidean) space, the static BPS branes look like (-1) branes, *i.e.* instantons.
- After this worldvolume dimensional reduction, the theory contains transverse-space gravity coupled to a nonlinear sigma model:

$$I_\sigma = \int d^D y \sqrt{g}\left(R - \tfrac{1}{2} G_{AB}(\phi)\partial_i\phi^A \partial_j \phi^B g^{ij}\right)$$

- The nonlinear sigma model has the structure G/H':

D	G	H'
9	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{SO}(1,1)$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2,1) \times \mathrm{SO}(1,1)$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(3,2)$
6	$\mathrm{SO}(5,5)$	$\mathrm{SO}(5,\mathbb{C})$
5	$\mathrm{E}_{6(+6)}$	$\mathrm{USP}(4,4)$
4	$\mathrm{E}_{7(+7)}$	$\mathrm{SU}^*(8)$
3	$\mathrm{E}_{8(+8)}$	$\mathrm{SO}^*(16)$

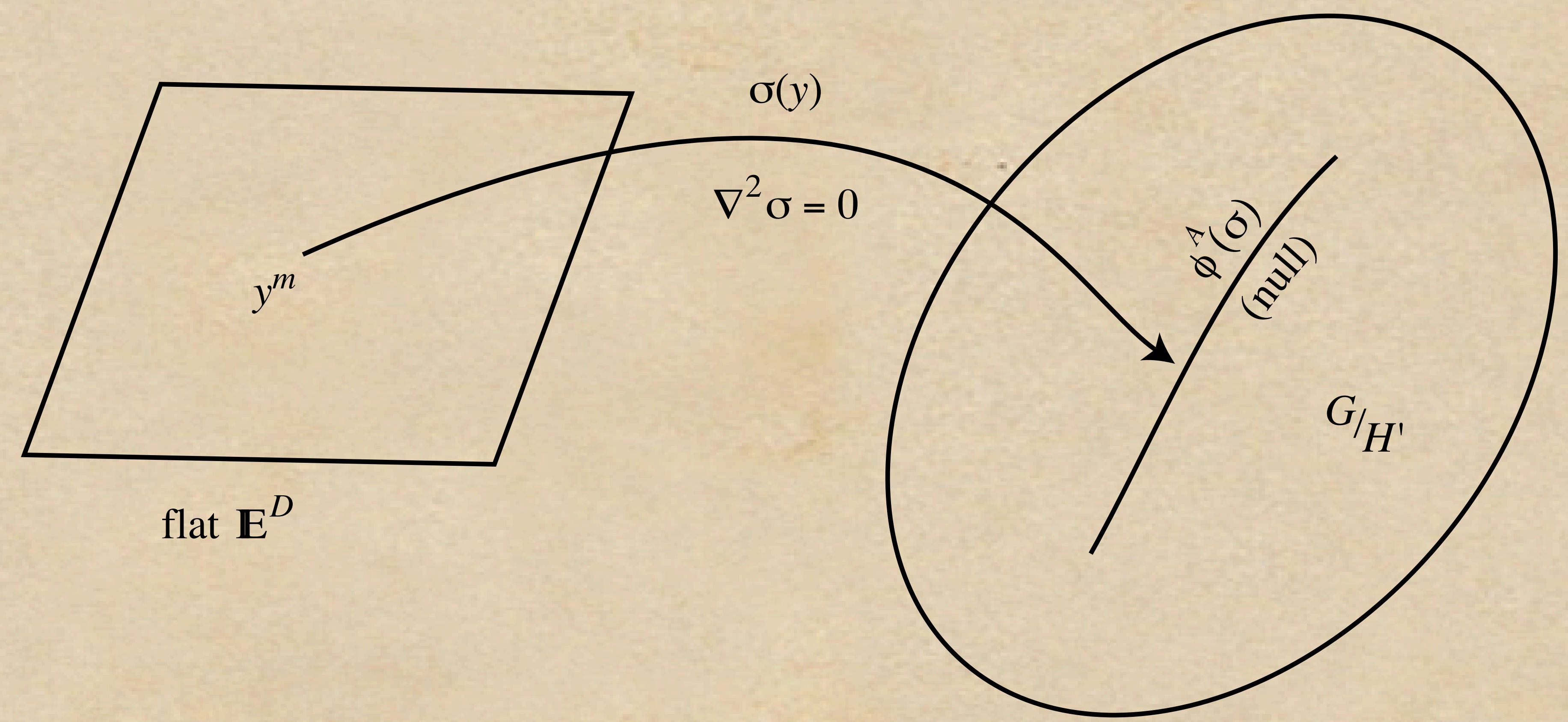
Note that H' is noncompact as a result of the reduction on the time coordinate. So the sigma-model metric G_{AB} is not positive definite, allowing for nontrivial instanton solutions in the Euclidean transverse space.

- The equations of motion for the reduced theory are

$$\frac{1}{\sqrt{g}}\nabla_i(\sqrt{g}g^{ij}G_{AB}(\phi)\partial_j\phi^B) = 0$$

$$R_{ij} = \frac{1}{2}G_{AB}(\phi)\partial_i\phi^A\partial_j\phi^B$$

- The standard family of BPS p-brane solutions corresponds to taking the reduced transverse-space metric to be flat, $g^{ij} = \delta^{ij}$. The equations of motion are then solved by taking the sigma-model fields to depend on the transverse coordinates y^m via a harmonic map: $\phi^A(y) = \phi^A(\sigma(y))$; $\nabla^2\sigma = 0$ where the curve $\phi^A(\sigma)$ in G/H' is null, *i.e.* $G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma} = 0$



A brane application: D=5 Horava-Witten theory

Lukas, Ovrut,
Stelle & D. Waldram,

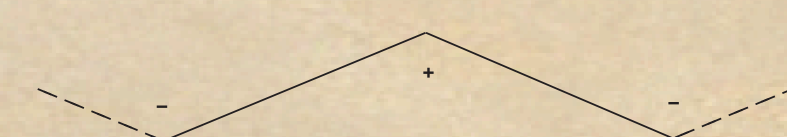
- Dimensionally reducing M-theory on a Calabi-Yau manifold with $G_{[4]}$ field-strength flux turned on yields a D=5 action which includes a potential for the breathing mode V that parametrizes the Calabi-Yau volume:

$$I^5_{\rm M} = -\frac{1}{2\kappa^2}\int d^5x\,[R + G_{ij}(b)\partial_\mu b^i\partial^\mu b^j + \tfrac{1}{2}V^{-2}\partial_\mu V\partial^\mu V$$
$$+\tfrac{1}{2}V^{-2}G^{ij}(b)\alpha_i\alpha_j + \text{more}]$$

- The resulting D=5 field equations admit a 3-brane:

$$ds^2_5 = e^{2A(y)}dx^\mu dx^\nu\eta_{\mu\nu} + e^{2B(y)}dy^2 \quad \mu\ ,\nu = 0,1,2,3$$
$$e^{A(y)} = \tilde{k}V^{\frac{1}{6}}(y) \qquad e^{2B(y)} = kV^{\frac{2}{3}}(y)$$
$$V(y) = (\tfrac{1}{6}d_{ijk}f^if^jf^k)^2 \qquad b^i(y) = V^{-\frac{1}{6}}f^i$$
$$d_{ijk}f^jf^k = H_i(y) \qquad H_i(y) = 2\sqrt{2}k\alpha_i|y| + k_i$$

Codimension-one harmonic function is linear; may be taken to be periodic with kinks to generate 2-brane Horava-Witten system.



D=5 reduction of IIB Theory

Bremer, Duff, Lü, Pope & Stelle

- Consider D=10 type IIB supergravity, keeping just the metric and the 5-form self-dual field strength:

$$R_{MN} = \tfrac{1}{96} F_{MPQRS} F_N{}^{PQRS}$$

$$F_{[5]} = {}^*F_{[5]} \qquad\qquad dF_{[5]} = 0$$

- The Kaluza-Klein ansatz for reduction on S^5 to D=5 is

$$ds^2_{10} = e^{2\alpha\varphi} ds^2_5 + e^{2\beta\varphi} ds^2(S^5)$$

$$\alpha = \tfrac{1}{4}\sqrt{\tfrac{5}{3}} \qquad\qquad \beta = -\tfrac{3}{5}\alpha$$

$$F_{[5]} = 4m e^{8\alpha\varphi} \epsilon_{[5]} + 4m \epsilon_{[5]}(S^5)$$

where φ is the "breathing mode" determining the local size of the S^5.

- The reduction to D=5 on S^5 yields the Lagrangian

$$\mathcal{L}_5 = eR - \tfrac{1}{2}e(\partial\varphi)^2 - 8m^2 e e^{8\alpha\varphi} + R_5 e e^{\frac{16}{5}\alpha\varphi}$$

where the m^2 exponential term arises from the 5-form flux and the R_5 term arises from the S^5 components of the D=10 Ricci scalar.

  - This two-exponential structure is characteristic of an S^5 Kaluza-Klein reduction.

  - A consequence of the two-exponential structure is the existence of an $AdS_5 \times S^5$ solution.

$$R_{\mu\nu} = -4m^2 e^{8\alpha\varphi_*} g_{\mu\nu} \qquad\qquad e^{\frac{24\alpha}{5}\varphi_*} = \frac{R_5}{20m^2}$$

Randall-Sundrum 3-brane

Duff, Liu & Stelle
Cvetič, Lü & Pope

- The D=5 IIB theory admits a 1/2 supersymmetric 3-brane solution

$$
\begin{aligned}
ds_5^2 &= e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} dy^2 \\
e^{-\frac{7}{\sqrt{15}}\varphi} &= H = k|y| + c \qquad B = -4A \\
e^{4A} &= b_1 H^{\frac{2}{7}} + b_2 H^{\frac{5}{7}}
\end{aligned}
$$

where $b_1 = \pm 28m/(3k)$, $\quad b_2 = \pm 14/(15k)\sqrt{5R_5}$

- In writing the above "two-sided" domain-wall solution, strictly speaking, one needs to modify the reduction ansatz for the D=10 5-form in order to incorporate the Z_2 symmetry:

$$
F_{[5]} = 4m\theta(y)e^{8\alpha\varphi}\epsilon_{[5]} + 4m\theta(y)\epsilon_{[5]}(S^5)
$$

- Obtaining the pure AdS_5 form of the Randall-Sundrum construction involves taking a $k \to 0$ limit in the 3-brane solution. Then, after a coordinate transformation one obtains the Poincaré-coordinate form of AdS_5:

$$ds_5^2 = e^{\frac{-2|z|}{L_{\rm AdS}}} dx^\mu dx^\nu \eta_{\mu\nu} + dz^2$$

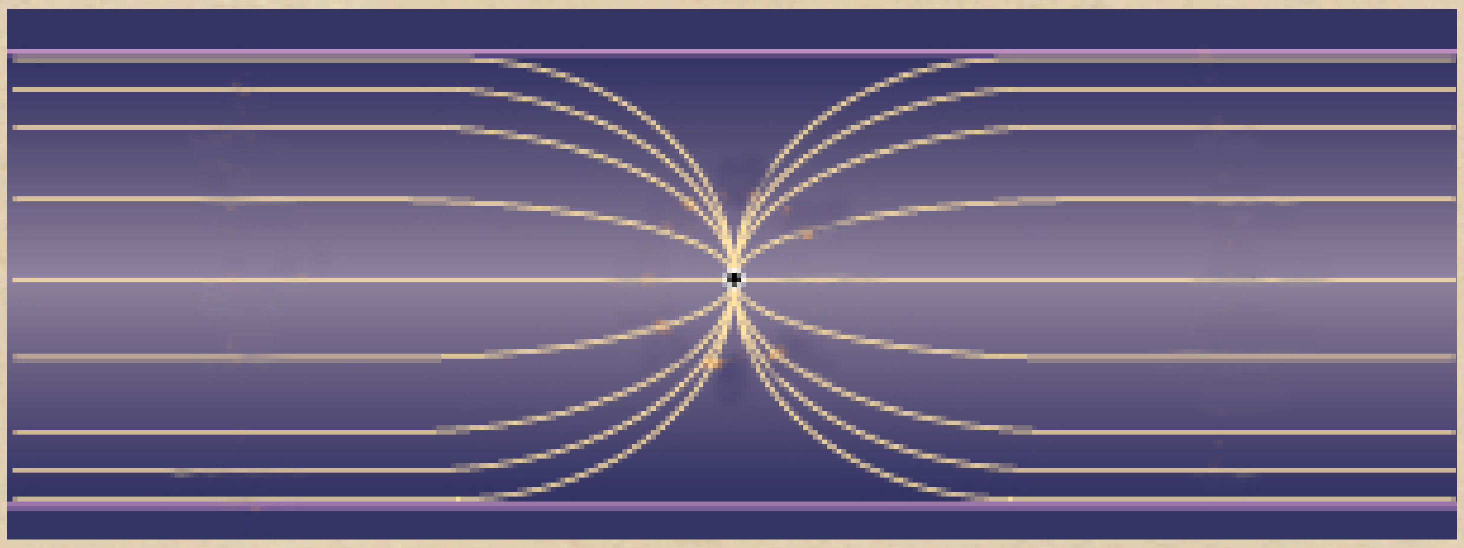
- The scale parameter of the AdS_5 space is related to the original 3-brane parameters by

$$L_{\rm AdS} = m^{-1} \left(\frac{20m^2}{R_5} \right)^{\frac{5}{6}}$$

- Putting kinks in the harmonic function/patching slices of AdS_5 space gives single or multi-brane Randall-Sundrum configurations.
- From the D=5 point of view, such situations are 1/2 supersymmetric.

- From the D=10, point of view, however, something rather strange happens: supersymmetry breaks. Liu & Sati

- The two-sided D=5 solution lifts to a two-world structure in D=10



- The D=10 singularity structure leads to inconsistent integrability conditions for Killing spinors in the 2-world D=10 solution. Kalkkinen, Lehners, Smyth & Stelle

- This leads to an interesting hierarchy of fermion masses for fluctuations about the brane background: bulk fermions have masses $\sim (L_{\mathrm{AdS}})^{-1}$ while fermions localized on a single RS1 end-of-world brane have masses

$$m_{\text{brane fermion}} \sim L_{\mathrm{AdS}}^{-1} e^{-\left(\frac{\ell_{\text{orbifold}}}{L_{\mathrm{AdS}}}\right)}$$

cf. Bagger & Belyaev