SUPERSTRING REALIZATIONS OF SUPERGRAVITY IN TEN AND LOWER DIMENSIONS

John H. Schwarz

Dedicated to the memory of Joël Scherk

SOME FAMOUS SCHERK PAPERS

Dual Models For Nonhadrons

J. Scherk, J. H. Schwarz

Nucl. Phys. B81, 118, 1974

Supersymmetric Yang-Mills Theories

L. Brink, J. H. Schwarz, J. Scherk

Nucl. Phys. B121, 77, 1977

Supersymmetry, Supergravity Theories and the Dual Spinor Model

F. Gliozzi, J. Scherk, D. I. Olive

Nucl. Phys. B122, 253, 1977

Supergravity Theory in Eleven-Dimensions

E. Cremmer, B. Julia, J. Scherk

Phys. Lett. B76, 409, 1978

Spontaneous Breaking of Supersymmetry through Dimensional Reduction

J. Scherk, J. H. Schwarz

Phys. Lett. B82, 60, 1979

How to Get Masses from Extra Dimensions

J. Scherk, J. H. Schwarz

Nucl. Phys. B153, 61-88, 1979

A more accurate title for this talk is

SUPERSTRING REALIZATIONS OF SUPERGRAVITY
IN TEN AND ELEVEN DIMENSIONS

This is a review talk, which contains no new results. The material that follows is extracted from Chapter 8 of:

String Theory and M-Theory: A Modern Introduction
Katrin Becker, Melanie Becker, and John H. Schwarz

This will be published soon by Cambridge University Press.

1 Eleven-dimensional supergravity

The bosonic part of the 11-dimensional supergravity action is

$$2\kappa_{11}^2 S = \int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4,$$

where R is the scalar curvature, $F_4 = dA_3$ is the field strength associated with the potential A_3. The relation between the 11-dimensional Newton's constant G_{11}, the gravitational constant κ_{11}, and the 11-dimensional Planck length $\ell_{\rm p}$ is

$$16\pi G_{11} = 2\kappa_{11}^2 = \frac{1}{2\pi}(2\pi\ell_{\rm p})^9.$$

The quantity $|F_4|^2$ is defined by the rule

$$|F_n|^2 = \frac{1}{n!} G^{M_1N_1} G^{M_2N_2} \cdots G^{M_nN_n} F_{M_1M_2\cdots M_n} F_{N_1N_2\cdots N_n}.$$

The complete action of 11-dimensional supergravity is invariant under the local supersymmetry transformations

$$\begin{aligned}
\delta E_M^A &= \bar{\varepsilon}\Gamma^A \Psi_M, \\
\delta A_{MNP} &= -3\bar{\varepsilon}\Gamma_{[MN}\Psi_{P]}, \\
\delta \Psi_M &= \nabla_M \varepsilon + \tfrac{1}{12}\left(\Gamma_M \mathbf{F}^{(4)} - 3\mathbf{F}_M^{(4)}\right)\varepsilon.
\end{aligned}$$

The formula for $\delta\Psi_M$ displays the terms that are of leading order in fermi fields. We have introduced the definitions

$$\mathbf{F}^{(4)} = \frac{1}{4!} F_{MNPQ}\Gamma^{MNPQ} \quad \text{and} \quad \mathbf{F}_M^{(4)} = \frac{1}{3!} F_{MNPQ}\Gamma^{NPQ}.$$

M-branes

An important feature of M-theory (and 11-dimensional supergravity) is the presence of the three-form gauge field A_3. It can couple electrically to a two-brane, called the M2-brane, and magnetically to a five-brane, called the M5-brane.

M-branes are BPS branes, whose tensions can be computed exactly. The results are

$$T_{\rm M2} = 2\pi(2\pi\ell_{\rm p})^{-3} \qquad \text{and} \qquad T_{\rm M5} = 2\pi(2\pi\ell_{\rm p})^{-6}.$$

2 Type IIA supergravity

The action of 11-dimensional supergravity is related to the actions of the various ten-dimensional supergravity theories, which are the low-energy effective descriptions of superstring theories.

The most direct connection is between 11-dimensional supergravity and type IIA supergravity. The deep reason is that M-theory compactified on a circle of radius R corresponds to type IIA superstring theory in ten dimensions with coupling constant $g_{\rm s} = R/\sqrt{\alpha'}$.

Type IIA supergravity can be obtained from 11-dimensional supergravity by *dimensional reduction*, *i.e.*, only keeping the zero modes in the Fourier expansions on the circle.

Bosonic fields

Dimensional reduction of the metric gives

$$G_{MN} = e^{-2\Phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\Phi} A_\mu A_\nu & e^{2\Phi} A_\mu \\ e^{2\Phi} A_\nu & e^{2\Phi} \end{pmatrix},$$

where the fields depend on the ten-dimensional space-time coordinates x^μ only. One gets a ten-dimensional metric $g_{\mu\nu}$, a $U(1)$ gauge field A_μ, and a scalar dilaton field Φ.

In terms of the inverse elfbein, the reduction is given by

$$E_A^M = \begin{pmatrix} e^{\Phi/3} e_a^\mu & 0 \\ -e^{\Phi/3} A_a & e^{-2\Phi/3} \end{pmatrix}.$$

The three-form in $D = 11$ gives rise to a three-form and a two-form in $D = 10$

$$A^{(11)}_{\mu\nu\rho} = A_{\mu\nu\rho} \qquad \text{and} \qquad A^{(11)}_{\mu\nu 11} = B_{\mu\nu},$$

with the corresponding field strengths given by

$$F^{(11)}_{\mu\nu\rho\lambda} = F_{\mu\nu\rho\lambda} \qquad \text{and} \qquad F^{(11)}_{\mu\nu\rho 11} = H_{\mu\nu\rho}.$$

Dimensional reduction can lead to somewhat lengthy formulas due to the nondiagonal form of the metric. A useful trick for dealing with this is to convert to tangent-space indices, since the reduction of the tangent-space metric is trivial.

With this motivation, consider

$$F^{(11)}_{ABCD} = E^M_A E^N_B E^P_C E^Q_D F^{(11)}_{MNPQ}.$$

There are two cases depending on whether the indices (A, B, C, D) are all ten-dimensional or one of them is 11:

$$F^{(11)}_{abcd} = e^{4\Phi/3}(F_{abcd} + 4A_{[a}H_{bcd]}) = e^{4\Phi/3}\widetilde{F}_{abcd},$$

$$F^{(11)}_{abc11} = e^{\Phi/3}H_{abc}.$$

It follows that upon dimensional reduction the 11-dimensional field strength is a combination of a four-form and a three-form field strength

$$\mathbf{F}^{(4)} = e^{4\Phi/3}\widetilde{\mathbf{F}}^{(4)} + e^{\Phi/3}\mathbf{H}^{(3)}\Gamma_{11},$$

where Γ_{11} is the ten-dimensional chirality operator. The quantities $\widetilde{\mathbf{F}}^{(4)}$ and $\mathbf{H}^{(3)}$ are defined in the same way as $\mathbf{F}^{(4)}$.

In ten dimensions

$$16\pi G_{10} = 2\kappa_{10}^2 = \frac{1}{2\pi}(2\pi\ell_{\rm s})^8 g_{\rm s}^2.$$

Dimensional reduction on a circle of radius R_{11} gives a relation between Newton's constant in ten and 11 dimensions

$$G_{11} = 2\pi R_{11} G_{10}.$$

One deduces that the radius of the circle is

$$R_{11} = g_{\rm s}^{2/3}\ell_{\rm p} = g_{\rm s}\ell_{\rm s}.$$

Type IIA Action

The bosonic action in the *string frame* for type IIA supergravity is obtained from the zero-mode truncation of the bosonic $D = 11$ action. The result contains three distinct types of terms

$$S = S_{\rm NS} + S_{\rm R} + S_{\rm CS}.$$

The first term is

$$S_{\rm NS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\, e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right).$$

This string-frame action is characterized by the exponential dilaton dependence in front of the curvature scalar. By a Weyl rescaling of the metric, this action can be transformed to the *Einstein frame* in which the Einstein term has the conventional form.

The remaining two terms in the action S involve the R–R fields and are given by

$$S_{\mathrm{R}} = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(|F_2|^2 + |\widetilde{F}_4|^2\right),$$

$$S_{\mathrm{CS}} = -\frac{1}{4\kappa^2}\int B_2 \wedge F_4 \wedge F_4.$$

3 Type IIB supergravity

The guiding principles to construct this theory come from supersymmetry as well as gauge invariance. One challenging feature of the type IIB theory is that it contains a self-dual five-form field strength. This introduces an obstruction to formulating the action in a manifestly covariant form. One strategy for dealing with this is to focus on the field equations instead, since they can be written covariantly.

Field content

The type IIB supergravity spectrum consists of

- **Fermions**: two left-handed Majorana–Weyl gravitinos and two right-handed Majorana–Weyl dilatinos
- **NS–NS bosons**: the metric (or zehnbein), the two-form B_2, and the dilaton Φ.
- **R–R bosons**: antisymmetric tensor fields C_0, C_2, and C_4. The latter has a self-dual field strength $\widetilde{F}_5$.

Global $SL(2,\mathbb{R})$ symmetry

Type IIB supergravity has $SL(2,\mathbb{R})$ global symmetry. The theory has two two-form potentials, B_2 and C_2, which transform as a doublet under the $SL(2,\mathbb{R})$ symmetry group.

The complex scalar field τ, defined by

$$\tau = C_0 + ie^{-\Phi},$$

is useful because it transforms nonlinearly by the familiar rule

$$\tau \to \frac{a\tau + b}{c\tau + d}.$$

The field C_0 is sometimes referred to as an axion, because of the shift symmetry $C_0 \to C_0 + \text{constant}$. The field τ is then referred to as an *axion–dilaton field*.

Type IIB S-duality

The global $SL(2, \mathbb{R})$ symmetry of type IIB supergravity is not shared by the full type IIB superstring theory. It is broken by various stringy and quantum effects to the discrete subgroup $SL(2, \mathbb{Z})$. The transformation $\tau \to -1/\tau$ inverts the coupling for $C_0 = 0$. This is the S-duality transformation.

The full group $SL(2, \mathbb{Z})$ is called the U-duality group.

(p,q) strings

Since there are two two-form gauge fields B_2 and C_2 there are two types of charge that a string can carry. The F-string (or fundamental string) has charge $(1, 0)$, which means that it has one unit of the charge that couples to B_2 and none of the charge that couples to C_2. In similar fashion, the D-string couples to C_2 and has charge $(0, 1)$.

Since the two-forms form a doublet of $SL(2, \mathbb{R})$ it follows that these strings also transform as a doublet. In general, they transform into (p, q) strings, which carry both kinds of charge. The restriction to the $SL(2, \mathbb{Z})$ subgroup is essential to ensure that these charges are integers, as is required by the Dirac quantization conditions.

The (p, q) strings are all on an equal footing. This implies that each of their tensions saturates a BPS bound given by supersymmetry, and this uniquely determines what their tensions are.

In the string frame, the tensions are

$$T_{(p,q)} = |p - q\tau_{\mathrm{B}}|T_{\mathrm{F1}} = T_{\mathrm{F1}}\sqrt{\left(p - q\frac{\theta_0}{2\pi}\right)^2 + \frac{q^2}{g_s^2}},$$

where we have defined the vev

$$\tau_{\mathrm{B}} = \langle\tau\rangle = \langle C_0 + ie^{-\Phi}\rangle = \frac{\theta_0}{2\pi} + \frac{i}{g_{\mathrm{s}}}$$

and

$$T_{\mathrm{F1}} = T_{(1,0)} = \frac{1}{2\pi\ell_{\mathrm{s}}^2}.$$

4 An M-theory/type IIB superstring duality

M-theory compactified on a circle gives the type IIA superstring theory. Furthermore, by T-duality, type IIA superstring theory on a circle corresponds to type IIB superstring theory on a dual circle.

Putting these two facts together, there should be a duality between M-theory on a two-torus T^2 and type IIB superstring theory on a circle S^1. The M-theory torus is characterized by an area $A_{\rm M}$ and a modulus $\tau_{\rm M}$, while the IIB circle has radius $R_{\rm B}$.

Since all of the (p,q) strings in type IIB superstring theory are related by $SL(2,\mathbb{Z})$ transformations, they are all equivalent, and any one of them can be weakly coupled. However, when one is weakly coupled, all of the others are necessarily strongly coupled.

Let us consider an arbitrary (p, q) string and write down the spectrum of its nine-dimensional excitations in the limit of weak coupling using standard string theory formulas:

$$M_{\rm B}^2 = \left(\frac{K}{R_{\rm B}}\right)^2 + (2\pi R_{\rm B} W T_{(p,q)})^2 + 4\pi T_{(p,q)}(N_{\rm L} + N_{\rm R}).$$

K is the Kaluza–Klein excitation number and W is the string winding number. $N_{\rm L}$ and $N_{\rm R}$ are excitation numbers of left-moving and right-moving oscillator modes, and the level-matching condition is

$$N_{\rm R} - N_{\rm L} = KW.$$

The plan is to use the formula above for all the (p, q) strings simultaneously.

The formula is not correct at strong coupling, and at most one of the strings is weakly coupled. The appropriate trick in this case is to consider only BPS states, *i.e.*, ones belonging to short supersymmetry multiplets, since their mass formulas can be reliably extrapolated to strong coupling.

The BPS states are given by

$$N_{\rm L} = 0 \quad \text{or} \quad N_{\rm R} = 0.$$

In this way, one obtains exact mass formulas for all the BPS states in the spectrum – many more than appear in any perturbative limit.

There is a unique correspondence between the three integers W, p, q, where p and q are coprime, and an arbitrary pair of integers n_1, n_2

given by

$$(n_1, n_2) = (Wp, Wq).$$

The integer W is the greatest common divisor of n_1 and n_2.

Altogether, BPS states are characterized by three arbitrary integers (K, n_1, n_2) and oscillator excitations corresponding to $N_{\rm L} = |WK|$, tensored with a 16-dimensional short multiplet from the $N_{\rm R} = 0$ sector (or *vice versa*).

Let us now consider M-theory compactified on a torus. If the two periods in the complex plane, which define the torus, are $2\pi R_{11}$ and $2\pi R_{11}\tau_{\rm M}$, then

$$A_{\rm M} = (2\pi R_{11})^2 {\rm Im}\, \tau_{\rm M}$$

is the area of the torus. In terms of coordinates $z = x + iy$ on the torus, single-valued wave functions have the form

$$\psi_{n_1,n_2} \sim \exp\left\{\frac{i}{R_{11}}\left[n_2 x - \frac{n_2 \mathrm{Re}\,\tau_{\mathrm{M}} - n_1}{\mathrm{Im}\,\tau_{\mathrm{M}}} y\right]\right\}.$$

These characterize Kaluza–Klein excitations. The contribution to the mass-squared is given by the eigenvalue of $-\partial_x^2 - \partial_y^2$:

$$M_{\mathrm{KK}}^2 = \frac{1}{R_{11}^2}\left[n_2^2 + \frac{(n_2 \mathrm{Re}\,\tau_{\mathrm{M}} - n_1)^2}{(\mathrm{Im}\,\tau_{\mathrm{M}})^2}\right] = \frac{|n_1 - n_2 \tau_{\mathrm{M}}|^2}{(R_{11} \mathrm{Im}\,\tau_{\mathrm{M}})^2}.$$

This term has the right structure to match the type IIB string winding-mode terms, described above, for the identification

$$\tau_{\mathrm{M}} = \tau_{\mathrm{B}}.$$

The normalization of $M_{\rm KK}^2$ and the winding-mode contribution to $M_{\rm B}^2$ is not the same, because they are measured in different metrics. The matching determines how to relate the two metrics.

This identification implies that the nonperturbative $SL(2,\mathbb{Z})$ symmetry of type IIB superstring theory, after compactification on a circle, has a *dual M-theory interpretation as the modular group of a toroidal compactification*!

Modular transformations of the torus are symmetries, since they correspond to the disconnected components of the diffeomorphism group. Once the symmetry is established for finite $R_{\rm B}$, it should also persist in the decompactification limit $R_{\rm B} \to \infty$.

To go further requires an M-theory counterpart of the Kaluza–Klein term $(K/R_{\rm B})^2$ in the type IIB superstring mass formula. Here there is also a natural candidate: wrapping M-theory M2-branes so as to cover the torus K times.

If the M2-brane tension is $T_{\rm M2}$, this gives a contribution

$$(A_{\rm M} T_{\rm M2} K)^2$$

to the mass-squared. Matching the normalization of this term and the Kaluza–Klein term one learns that the compactification volumes $R_{\rm B}$ and A_M are related by

$$\frac{g_{\rm s}^2}{T_{\rm F1} R_{\rm B}^2} = T_{\rm M2} \left(\frac{A_M}{{\rm Im}\,\tau_M}\right)^{3/2}.$$

5 Conclusion

Joël Scherk's remarkable contributions in the decade of the 1970s set the stage for many of the exciting developments that followed. It is a great pity that he could not participate in these developments.

THE END