Generalised Dualities and Self-Dualities

30 Years of Supergravity

Institut Henri Poincaré
October 19'th, 2006

- Global Symmetries in Supergravities
- Rôle of Dualisation
- Consistent Sphere Reductions – Yes or No?
- Generalised Dualities
- Superdualities

Work with E. Cremmer, B. Julia and H. Lü

Global Symmetries from Toroidal Reduction

When any theory with general coordinate invariance is reduced on an n-dimensional torus, with $x^M \to (x^\mu, y^i)$, the residual general coordinate transformations

$$\delta y^i = -\Lambda^i{}_j\, y^j$$

for constant $\Lambda^i{}_j$ give global $GL(n,\mathbb{R})$ symmetries in the lower dimension.

The $SL(n,\mathbb{R})$ factor leaves the metric invariant, but in general the extra $\mathbb{R}$ factor scales the metric. It too can become a purely "internal" symmetry (leaving the lower-dimensional metric invariant) if the theory admits an overall scaling symmetry (at the level of the equations of motion).

Pure Einstein gravity has this symmetry: under $g_{\mu\nu} \to \lambda^2\, g_{\mu\nu}$,

$$\mathcal{L} = \sqrt{-g} R \longrightarrow \lambda^{D-2}\, \mathcal{L}\,.$$

$D = 11$ supergravity also has such a symmetry, under

$$g_{\mu\nu} \longrightarrow \lambda^2\, g_{\mu\nu}\,, \qquad A_{\mu\nu\rho} \longrightarrow \lambda^3\, A_{\mu\nu\rho}\,,$$

$$\mathcal{L} = \sqrt{-g}(R - \tfrac{1}{48}F^2) + \tfrac{1}{6} F\wedge F\wedge A \to \lambda^9\, \mathcal{L}$$

Extends to all dimensionally-reduced maximal supergravities.

Toroidal Reduction, Step-by-Step

A toroidal KK reduction can be broken up into succesive circle reductions. At each step, we have $\hat{x}^M \to (x^\mu, z)$ and for the metric

$$d\hat{s}^2 = e^{2\alpha\phi}\, ds^2 + e^{2\beta\phi}\, (dz + \mathcal{A})^2\,. \tag{1}$$

ϕ is the "dilaton" and $\mathcal{A}$ the KK vector (1-form). A p-form potential reduces according to

$$\hat{A}_{(p)} = A_{(p)} + A_{(p-1)} \wedge dz\,. \tag{2}$$

Iterating the 1-step reduction n times gives the T^n reduction.

The global symmetries can be characterised by the lower-dimensional scalar fields, which form a non-linear sigma coset model G/H with G the global symmetry group. Higher form fields transform as linear representations of G.

The scalar sector comprises n dilatonic scalars $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$, plus axionic scalars which are 0-form potentials coming from the reduction of the KK 1-forms, and from any other form fields in the higher-dimensional theory.

Pure Gravity on T^n

After one step of reduction, $\hat{\mathcal{L}} = \sqrt{-\hat{g}}\hat{R}$ reduces to

$$\mathcal{L} = \sqrt{-g}R - \tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{4}e^{b\phi_1}\mathcal{F}^2\,.$$

At the next step, the KK vector gives a vector and an axion, etc.

The scalar sector of the theory reduced on T^n comprises n dilatons $\vec{\phi}$, and $\frac{1}{2}n(n-1)$ axions $\chi^i{}_j$ (with $i<j$), coming from the reduction of KK 1-forms to axions. This gives $\frac{1}{2}n(n+1)$ scalar fields in the coset $GL(n,\mathbb{R})/O(n)$. The structure of the scalar Lagrangian is

$$\mathcal{L} = \sqrt{-g}\Big(-\tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,(\partial\chi^i{}_j + \cdots)^2\Big)\,.$$

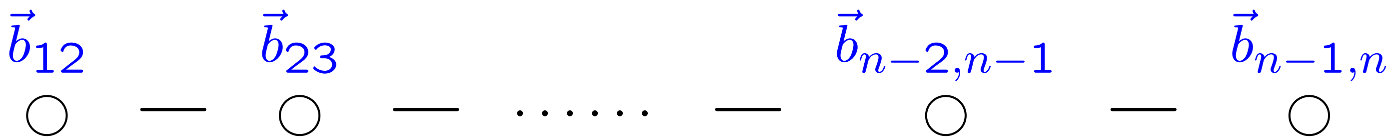
(The ellipses denote non-linear terms like $\chi^i{}_k\,\partial\chi^k{}_j$, etc.)

After extracting an overall volume scalar (the $\mathbb{R}$ factor in $GL(n,\mathbb{R})$), the "dilaton vectors" $\vec{b}_{ij}$ form the positive roots of the $SL(n,\mathbb{R})$ algebra. The simple roots are

$$\vec{b}_{12}\,,\qquad \vec{b}_{23}\,,\qquad \vec{b}_{34}\,,\qquad \ldots\,,\qquad \vec{b}_{n-1,n}\,.$$

We have the usual $\vec{b}_{12}+\vec{b}_{23}=\vec{b}_{13}$, etc.

The $SL(n,\mathbb{R})$ Dynkin Diagram



The dilaton vectors $b_{i,i+1}$ generate the Dynkin diagram of $SL(n,\mathbb{R})$

Borel Subalgebra Parameterisation of the Scalar Coset

There is an axionic scalar $\chi^i{}_j$ associated with each positive-root generator $E_i{}^j$ of $SL(n,\mathbb{R})$. The dilatons ϕ are associated with the Cartan generators $\vec{H}$. The Borel subalgebra of $SL(n,\mathbb{R})$ is given by

$$[\vec{H}, E_i{}^j] = \vec{b}_{ij}\, E_i{}^j\,, \qquad [E_i{}^j, E_k{}^\ell] = \delta_k^j E_i{}^\ell - \delta_i^\ell E_k{}^j\,.$$

The coset representative $\mathcal{V}$ of $SL(n,\mathbb{R})/O(n)$ is then given by

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \prod_{i<j} e^{\chi^i{}_j\, E_i{}^j}\,.$$

(With anti-lexical ordering of the axion terms. i.e. $\chi^1{}_2$ at the right, etc.) The scalar Lagrangian is then given by

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})\,, \qquad \mathcal{M} \equiv \mathcal{V}^{\mathsf{T}}\mathcal{V}\,.$$

(Correctly generates all the non-linear $\chi\partial\chi$, etc., modifications too.)

This makes the $SL(n,\mathbb{R})$ symmetry of pure gravity on T^n manifest in a very simple way, and which easily generalises to supergravity reductions.

The Kaluza-Klein vectors $\mathcal{A}^i_{(1)}$, with $1 \le i \le n$, are described by terms in the lower-dimensional Lagrangian of the form

$$\mathcal{L}_F = -\frac{1}{4} \sum_{i=1}^{n} e^{\vec{b}_i \cdot \vec{\phi}} \, (\mathcal{F}^i_{(2)})^2 \,,$$

with the dilaton vectors $\vec{b}_i$ forming the weight vectors of the n-dimensional representation of $SL(n,\mathbb{R})$. After some work, it can be shown that $SL(n,\mathbb{R})$ (and the extra $\mathbb{R}$ factor) is a global symmetry of the full lower-dimensional Lagrangian.

The global symmetry of the T^n-reduced theory is seen most easily in the scalar sector. The rest follows.

T^n Reduction of $D = 11$ Supergravity

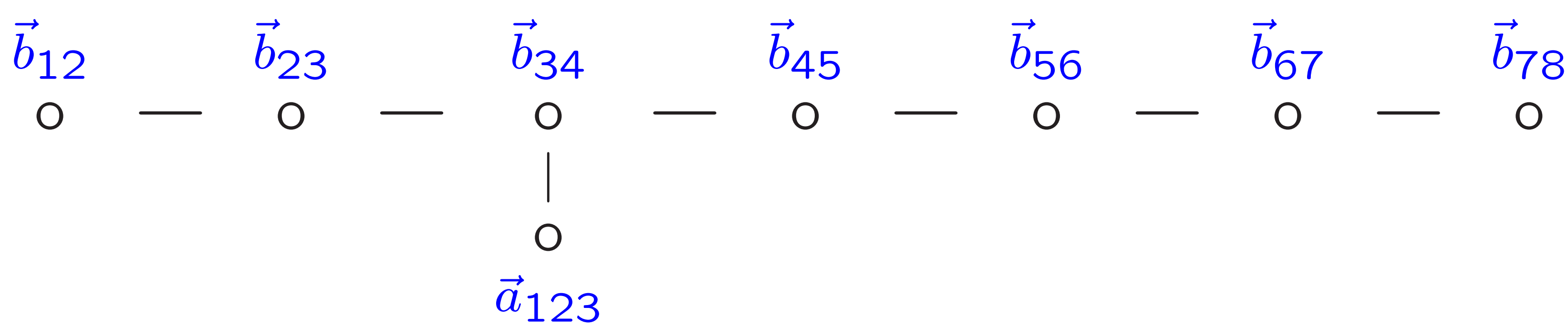
The T^n reduction of

$$\mathcal{L} = \sqrt{-\hat{g}}(\hat{R} - \tfrac{1}{48}\hat{F}_{(4)}^2) + \tfrac{1}{6}\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}$$

proceeds in a similar way. We get further form fields in the lower dimensions, coming from the reduction of $\hat{A}_{(3)}$. Things start to become interesting after reduction to $D \le 8$, since now there are extra scalars (axions) χ_{ijk} coming from $\hat{A}_{(3)}$. The scalar Lagrangian in the reduced theory is now of the form of the previous pure-gravity reduction, plus a part from the reduction of $\hat{A}_{(3)}$:

$$\mathcal{L} = -\tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}(\partial\chi^i{}_j + \cdots)^2 - \tfrac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}(\partial\chi_{ijk} + \cdots)^2 .$$

The dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ can again be interpreted as the positive roots of a Lie algebra, with simple roots $\vec{b}_{i,i+1}$ (as before), plus $\vec{a}_{123}$. They generate the E_n Dynkin diagram:



Global Symmetry in $D \geq 6$

In $D = 11 - n \geq 6$ dimensions, the symmetry enhancement to "E_n" is straightforward. We now introduce

$$\text{Cartan generators}: \quad \vec{H}$$
$$\text{Positive-root generators}: \quad E_i{}^j\,, \qquad E^{ijk}$$

for the dilatons $\vec{\phi}$ and the axions $\chi^i{}_j$ and χ_{ijk} respectively. They satisfy the algebra

$$\begin{aligned}
[\vec{H}, E_i{}^j] &= \vec{b}_{ij}\, E_i{}^j\,, \qquad [E_i{}^j, E_k{}^\ell] = \delta_k^j E_i{}^\ell - \delta_i^\ell E_k{}^j\,,\\
[\vec{H}, E^{ijk}] &= \vec{a}_{ijk}\, E^{ijk}\,, \qquad [E_\ell{}^m, E^{ijk}] = -3\delta_\ell^{[i}\, E^{jk]m}\,,\\
[E^{ijk}, E^{\ell mn}] &= 0\,.
\end{aligned}$$

We make the coset representative

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \prod_{i<j} e^{\chi^i{}_j\, E_i{}^j}\, e^{\sum_{i<j<k}\chi_{ijk}E^{ijk}}\,,$$

and find the scalar Lagrangian is produced (exactly) by

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})\,, \qquad \mathcal{M} \equiv \mathcal{V}^{\mathsf{T}}\mathcal{V}\,.$$

Global Symmetry in $D \le 5$

In $D = 11 - n \le 5$, the scalar sector described so far is incomplete. The generators $E_i{}^j$ and E^{ijk} are insufficient to fill out the positive-root space of E_n. We are missing 1 generator in $D = 5$, 7 in $D = 4$ and $36 = 28 + 8$ in $D = 3$.

The associated "missing" scalars arise from dualisation!

$$
\begin{aligned}
D = 5: &\quad A_{(3)} \longrightarrow \chi\,,\\
D = 4: &\quad A_{(2)i} \longrightarrow \chi_i\,,\\
D = 3: &\quad A_{(1)ij} \longrightarrow \chi_{ij}\,, \quad \text{and} \quad \mathcal{A}^i_{(1)} \longrightarrow \chi^i\,.
\end{aligned}
$$

With an appropriate augmentation of the set of generators, the algebra, and the coset representatives, we now get the correct scalar manifolds. For example, in $D = 5$ add a generator D, and now

$$
\begin{aligned}
[E^{ijk}, E^{\ell mn}] &= -\epsilon^{ijk\ell mn}\, D\,,\\
\mathcal{V} &= e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \prod_{i<j} e^{\chi^i{}_j\, E_i{}^j}\, e^{\sum_{i<j<k} \chi_{ijk} E^{ijk}}\, e^{\chi\, D}\,,
\end{aligned}
$$

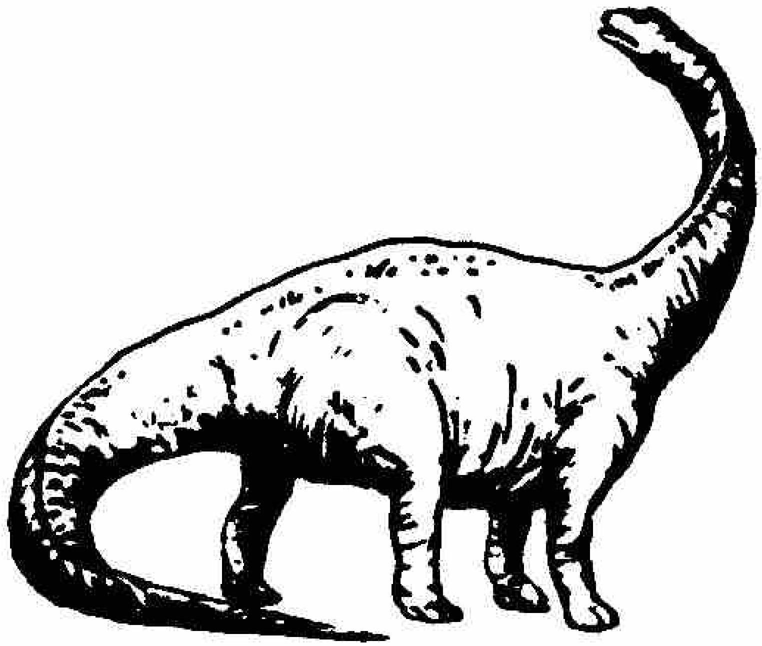
Further Remarks

- This procedure gives a simple derivation of the scalar coset G/H for reduction of $D = 11$ supergravity on T^n:

	G	H
$D = 10$	$O(1,1)$	1
$D = 9$	$GL(2,\mathbb{R})$	$O(2)$
$D = 8$	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$O(3) \times O(2)$
$D = 7$	$SL(5,\mathbb{R})$	$O(5)$
$D = 6$	$O(5,5)$	$O(5) \times O(5)$
$D = 5$	E_6	$USp(8)$
$D = 4$	E_7	$SU(8)$
$D = 3$	E_8	$O(16)$

- The enhanced global symmetries in $D \le 8$ require the FFA term in $D = 11$ with *exactly* the coefficient demanded by supersymmetry!

- The symmetry extends to the form fields (requires more work). So the full demonstration of the global symmetry is easy in $D = 10$, easy in $D = 3$ (all scalars), and much harder in the middle cases $4 \le D \le 6$.



"The brontosaurus is thin at one end,
much much thicker in the middle,
and thin at the other end."

Monty Python

Consistent Dimensional Reductions

- A *consistent* dimensional reduction is one where the "reduction ansatz" is substituted into the higher-dimensional equations of motion, yielding lower-dimensional equations of motion whose solutions imply solution of the higher-dimensional equations of motion.

- A consistent dimensional reduction can be performed on *any* internal space, provided that one keeps all the (infinite number) of massive as well as massless modes. Just a generalised Fourier expansion of the higher-dimensional theory.

- The interesting question is whether one can consistently truncate to the "massless sector," or at least to *some* finite subset of lower-dimensional fields. i.e. can one set the rest of the lower-dimensional fields to zero, *consistently with their equations of motion*?

- Group theory (truncation to singlets) guarantees this for reduction on S^1 or T^n (**Kaluza reduction – 1921**); and for group manifold reduction on G keeping $G_L \subset G_L \times G_R$ Yang-Mills (**De Witt reduction – 1963**).

- The most interesting cases are the "miraculous" ones where a consistent reduction on S^n, keeping all the $SO(n+1)$ Yang-Mills fields, can be performed. (**Pauli reduction – 1953**) Although Pauli envisaged such reductions, he recognised that they could not work in general, and in fact he had no example.

Consistent Sphere Reductions?

- Under what circumstances can there exist a consistent Pauli reduction on S^n, including the Yang-Mills fields of $SO(n+1)$ in a lower-dimensional theory with a finite number of fields (the "massless sector")?

- If such an S^n reduction is possible, then it will represent a *gauging* of the theory obtained by reduction on T^n.

- Conversely, if one turns off the gauge coupling in the S^n-reduced theory, by sending the radius to infinity, it should revert to the T^n reduction.

- For the S^n reduction to be a gauging of the T^n reduction, there must be at least an $SO(n+1)$ subgroup in the global symmetry group G of the T^n reduction.

- This immediately rules out the possibilty of a consistent Pauli reduction in any "generic" theory! (Which would have only $GL(n,\mathbb{R})$ global symmetry; this does not contain $SO(n+1)$.)

- Only in exceptional cases where there is a symmetry enhancement, is there any chance for a consistent Pauli reduction. For example, $D=11$ supergravity on S^4, or on S^7.

Doubled Formalism

- Global symmetries can depend on whether fields are dualised or not. Suggests it could be interesting to reformulate the theory so that the fields **and** their duals are all present.

- In eleven dimensions, $\mathcal{L} = R{*}\mathbf{1} - \frac{1}{2}{*}F_{(4)} \wedge F_{(4)} - \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)}$. Equation for $A_{(3)}$ is $d{*}F_{(4)} + \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0$.

- This implies $d({*}F_{(4)} + \frac{1}{2} A_{(3)} \wedge F_4) = 0$, so we can write the original equation of motion as the first-order equation

$$
{*}F_{(4)} = \widetilde{F}_{(7)} \equiv d\widetilde{A}_{(6)} - \tfrac{1}{2} A_{(3)} \wedge F_{(4)} \,.
$$

This equation is invariant under the infinitesimal gauge transformations

$$
\delta A_{(3)} = \Lambda_{(3)} \,, \qquad \delta \widetilde{A}_{(6)} = \widetilde{\Lambda}_{(6)} - \tfrac{1}{2} \Lambda_{(3)} \wedge A_{(3)} \,,
$$

where $d\Lambda_{(3)} = d\widetilde{\Lambda}_{(6)} = 0$.

- Gauge transformations now non-Abelian:

$$
\begin{aligned}
[\delta_{\Lambda_{(3)}}, \delta_{\Lambda'_{(3)}}] &= \delta_{\widetilde{\Lambda}''_{(6)}} \,, \qquad \widetilde{\Lambda}''_{(6)} \equiv \Lambda_{(3)} \wedge \Lambda'_{(3)} \,, \\
[\delta_{\Lambda_{(3)}}, \delta_{\widetilde{\Lambda}_{(6)}}] &= 0 \,, \qquad [\delta_{\widetilde{\Lambda}_{(6)}}, \delta_{\widetilde{\Lambda}'_{(6)}}] = 0 \,.
\end{aligned}
$$

Superalgebra of Doubled Formalism

- Introduce generators V for $\Lambda_{(3)}$ and $\widetilde{V}$ for $\widetilde{\Lambda}_{(6)}$. These satisfy the Lie superalgebra

$$\{V,V\} = -\widetilde{V}\,, \qquad [V,\widetilde{V}] = 0\,, \qquad [\widetilde{V},\widetilde{V}] = 0\,.$$

(Even (odd) generators for even (odd) forms Λ.)

- $\widetilde{V}$ is even, and can be diagonalised. For each eigenvalue there is a Clifford algebra in one generator. It is a Grassmann superalgebra deformed to a Clifford algebra because of the Chern-Simons term:

Grassmann + Chern-Simons = Clifford

- Introduce a "coset representative" $\mathcal{V} = e^{A_{(3)}V}\, e^{\widetilde{A}_{(6)}\widetilde{V}}$, so

$$\begin{aligned}\mathcal{G} \equiv d\mathcal{V}\,\mathcal{V}^{-1} &= dA_{(3)}V + (d\widetilde{A}_{(6)} - \tfrac{1}{2}A_{(3)} \wedge F_{(4)})\widetilde{V} \\ &= F_{(4)}V + \widetilde{F}_{(7)}\widetilde{V}\,.\end{aligned}$$

- The gauge transformations can now be expressed as

$$\mathcal{V}' = \mathcal{V}\, e^{\Lambda_{(3)}V}\, e^{\widetilde{\Lambda}_{(6)}\widetilde{V}}\,.$$

Twisted Self-Duality

- The equation of motion ($*F_{(4)} = \widetilde{F}_{(7)}$) can be written as a "twisted self-duality equation"

$$*\mathcal{G} = \mathcal{S}\,\mathcal{G}\,,$$

  where $\mathcal{S}$ is a pseudo-involution that maps between V and $\widetilde{V}$:

$$\mathcal{S}\,V = \widetilde{V}\,, \qquad \mathcal{S}\,\widetilde{V} = -V\,.$$

  (So $\mathcal{S}^2 = -1$.)

- Since $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$, we have the Cartan-Maurer equation

$$d\mathcal{G} = \mathcal{G} \wedge \mathcal{G}\,.$$

- Another formulation of the equation of motion is to start with $\mathcal{G} \equiv F_{(4)} V + \widetilde{F}_{(7)} \widetilde{V}$, and plug into the Cartan-Maurer equation. Since we then have

$$\mathcal{G} \wedge \mathcal{G} = -\tfrac{1}{2} F_{(4)} \wedge F_{(4)}\,\widetilde{V}\,,$$

  we indeed recover the equation of motion from the M-C equation.

Superdualities Maximal Supergravities

- Lower-dimensional gauged supergravities give larger superduality groups. Duals for all fields (except the metric) are now included.
- In D dimensions the algebra is a deformation of $G \ltimes G^*$, where G is itself the semi-direct product of the Borel subalgebra of the superalgebra $SL(11 - D|1)$ and a rank-3 tensor rep. G^* is the co-adjoint representation of G.
- Type IIB Supergravity is an unusual case for which the "superduality algebra" is purely bosonic. This is because all the fields (gauge potentials) are even-degree forms, and so the generators are all even.
- Possible extensions: Fermions; Include metric; E_9, E_{10}, E_{11},...
- Symmetries present already in $D = 11$? Or is that "slipping the rabbit into the hat"?