Super-gravitational waves, maximal and reduced supersymmetry

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30 years of Supergravity

Paris, October 2006

SUPERSYMMETRIC SOLUTIONS

Many applications

- M-theory
- String theory, duality
- Branes, Black holes
- Compactifications
- Spinorial geometry, special geometric structures
- AdS/CFT, gravity/Yang-Mills correspondences

ASKING THE RIGHT QUESTION!

- Intersecting M-branes (1996)

  with P.K.Townsend

  Are there (completely) localized brane intersections?

- Cargese (1999)

  Can the 1/2-susy solutions of D=11 and IIB supergravities, $N = 16$, be classified?

- H. Poincaré Institute (2000-01)

  Can the maximal susy solutions of D=11 and IIB supergravities, $N = 32$, be classified?

KILLING SPINOR EQUATIONS (KSE)

A parallel transport equation for the supercovariant connection $\mathcal{D}$

$$\delta\psi_A| = \mathcal{D}_A\epsilon = \nabla_A\epsilon + \Sigma_A(e,F)\epsilon = 0$$

and possibly algebraic equations

$$\delta\lambda| = \mathcal{A}(e,F)\epsilon = 0$$

where ∇ is the Levi-Civita connection, $\Sigma(e,F)$ a Clifford algebra element

$$\Sigma(e,F) = \sum_k \Sigma_{[k]}(e,F)\Gamma^{[k]}$$

e frame and F fluxes, ϵ spinor, Γ gamma matrices.

- N no of linearly independent solutions ϵ.

Can the KSE be solved without any assumptions on the metric and fluxes?

MAXIMAL SUPERSYMMETRY
D=11 supergravity

Cremmer, Julia, Scherk

The maximally susy solutions of D=11 SUGRA, $N = 32$, are locally isometric

$\mathbb{R}^{10,1}$ $\quad AdS_4 \times S^7 \quad AdS_7 \times S^4 \quad CW_{11}$

Proof: J. Figueroa-O'Farrill, GP

$$\mathcal{D}\epsilon = 0 \Rightarrow \mathcal{R}\epsilon := [\mathcal{D}, \mathcal{D}]\epsilon = 0 \Rightarrow \mathcal{R} = 0$$

$\mathcal{R} = 0$ implies

$$\nabla R = 0 \ , \quad \nabla F = 0$$

and the Plücker relation

$$i_X i_Y i_Z F \wedge F = 0$$

M is a symmetric space and F is a simple form.

- $F = 0$ $(\mathbb{R}^{10,1})$
- F timelike $(AdS_4 \times S^7)$
- F spacelike $(AdS_7 \times S^4)$

Freund Rubin

- F null (CW_{11})

Kowalski-Glikman

where

$$ds^2(AdS_{n+2}) = z^{-2}(dz^2 + ds^2(\mathbb{R}^{n,1}))$$
$$ds^2(CW_{n+2}) = 2dv(du - A_{ij}x^i x^j dv) + \sum_{i=1}^{n}(dx^i)^2$$

and A constant symmetric matrix.

IIB supergravity

Schwarz, West, Howe

The maximally susy solutions of IIB SUGRA, $N = 32$, are locally isometric

$$\mathbb{R}^{9,1} \quad AdS_5 \times S^5 \quad CW_{10}$$

Proof: J. Figueroa-O'Farrill, GP

$$\mathcal{A}\epsilon = 0 \Rightarrow \mathcal{A} = 0 \Rightarrow P = G = 0$$

In addition $\mathcal{R}\epsilon = \mathcal{R} = 0$ implies

$$\nabla R = 0 \,, \quad \nabla F^+ = 0$$

and the modified Plücker relation

$$i_X i_Y i_Z (F^+)^A \wedge F^+{}_A = 0$$

M is a symmetric space and F^+ is sum of two orthogonal simple forms.

- $F^+ = 0$ $(\mathbb{R}^{10,1})$
- F^+ non-null $(AdS_5 \times S^5)$

Schwarz

- F^+ null (CW_{10})

Blau, Hull, Figueroa-O'Farrill, GP

Penrose limits of $AdS_{p+1} \times S^q$ are either plane waves CW_{p+q+1} or Minkowski space $\mathbb{R}^{p+q,1}$

Blau, Hull, Figueroa-O'Farrill, GP

Strings can be solved on CW_{10}

Metsaev, Tseytlin

New tests for AdS/CFT

Berenstein, Maldacena, Nastase

REDUCED SUPERSYMMETRY

Holonomy

Hull, Duff, Liu, Tsimpis, GP

For generic D=11 and IIB backgrounds

$$\mathrm{hol}(\mathcal{D}) \subseteq SL(32, \mathbb{R})$$

because $\mathcal{R}$ takes values in $\mathfrak{sl}(32, \mathbb{R})$

For N-susy backgrounds

$$\begin{aligned}\mathrm{hol}(\mathcal{D}) &\subseteq SL(32-N, \mathbb{R}) \ltimes \oplus_N \mathbb{R}^{32-N} \\ &= \mathrm{Stab}(\epsilon) \subset SL(32, \mathbb{R})\end{aligned}$$

The consequences are

- There may be backgrounds for any N, however see preons ($N = 31$)
- Any subbundle $\mathcal{K}$ of the Spin bundle $\mathcal{S}$ can be Killing

Gauge Symmetry G

The gauge symmetry G of the KSE are the (local) transformations such that

$$g^{-1}\mathcal{D}(e,F)g=\mathcal{D}(e^g,F^g)$$

D=11 SUGRA: $G=Spin(10,1)$

IIB SUGRA: $G=Spin(9,1)\times U(1)$

- Backgrounds related by a gauge transformation are identified
- The geometry of backgrounds is (non-uniquely) characterized by the stability subgroup stab(ϵ) of the KS in G
- $G\subset\subset$ hol($\mathcal{D}$), e.g. 2 generic spinors in D=11 and IIB have stab(ϵ) $=\{1\}$

- For one spinor

  D=11: stab = $SU(5)$, $Spin(7) \ltimes \mathbb{R}^9$

  Bryan, Figueroa-O'Farrill

  IIB: stab = $Spin(7) \ltimes \mathbb{R}^8$, $SU(4) \ltimes \mathbb{R}^8$, G_2

Can extended gauge symmetries help?

Spin(9,1) SPINORS

Consider $U = \mathbb{C} < e_1, \dots, e_5 >$, $e_1, \dots, e_5$ orthonormal w.r.t $<,>$.

Dirac spinors: $\Delta_c = \Lambda^*(U)$

Weyl Spinors: $\Delta_c^+ = \Lambda^{\mathrm{ev}}(U)$, $\Delta_c^- = \Lambda^{\mathrm{od}}(U)$.

Gamma matrices on Δ_c:

$$
\begin{aligned}
\Gamma_0 \eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta \ , \\
\Gamma_5 \eta &= e_5 \wedge \eta + e_5 \lrcorner \eta \\
\Gamma_i \eta &= e_i \wedge \eta + e_i \lrcorner \eta \ , \qquad i = 1, \dots, 4 \\
\Gamma_{5+i} \eta &= i e_i \wedge \eta - i e_i \lrcorner \eta \ .
\end{aligned}
$$

The Dirac inner product:

$$D(\eta, \theta) = < \Gamma_0 \eta, \theta >$$

A Majorana inner product:

$$B(\eta, \theta) = < B(\eta^*), \theta > \ , \qquad B = \Gamma_{06789}$$

The Majorana reality condition can be chosen as

$$\eta = -\Gamma_0 B(\eta^*) = \Gamma_{6789}\eta^* \ .$$

$C = \Gamma_{6789}$ is the charge conjugation matrix.

Example

For Weyl spinor $a1 + be_{1234}$, $a, b \in \mathbb{C}$, the reality condition gives

$$\eta = a1 + a^* e_{1234} \ .$$

Two Majorana spinors: $1 + e_{1234}$ and $i1 - ie_{1234}$.

- $\mathrm{stab}(1 + e_{1234}) = Spin(7) \ltimes \mathbb{R}^8$
- $\mathrm{stab}(1+e_{1234}, i(1-e_{1234})) = SU(4) \ltimes \mathbb{R}^8$
- Δ_c has an oscillator basis, $\mu = 0, 1, \ldots, 4$

  $1, \quad e_\mu = e_\mu \wedge 1, \quad e_{\mu\nu} = e_\mu \wedge e_\nu \wedge 1, \quad \ldots$

SPINORIAL GEOMETRY

Gillard, Gran, GP

The ingredients of the spinorial method to classify supergravity backgrounds are

- Gauge symmetry of KSE

  Effective for backgrounds with small and large number of susies

- Spinors in terms of forms

  Convenient notation

- An oscillator basis in the space of spinors

  Allows to extract the geometric information from the KSE

HETEROTIC SUPERGRAVITY

Geometry of susy backgrounds has been investigated before

Strominger, Hull

In this case $\mathcal{D} = \hat{\nabla} = \nabla - \frac{1}{2}H$, H torsion, and

$$\mathrm{hol}(\mathcal{D}) = G = Spin(9,1)$$

In addition

$$\hat{\nabla}\epsilon = 0 \Rightarrow \hat{R}\epsilon = 0$$

So either

$$\hat{R} = 0$$

and M is a group Manifold ($dH = 0$), or

$$\mathrm{stab}(\epsilon) \neq \{1\}$$

- The parallel spinors can be chosen to be constant as in the Berger case for Riemannian manifolds

stab(ϵ)	N = 1	N = 2	N = 3	N = 4	N = 8	N = 16
$Spin(7) \ltimes \mathbb{R}^8$	$\surd$	-	-	-	-	-
$SU(4) \ltimes \mathbb{R}^8$	-	$\surd$	-	-	-	-
G_2	-	$\surd$	-	-	-	-
$Sp(2) \ltimes \mathbb{R}^8$	-	-	$\surd$	-	-	-
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	-	-	-	$\surd$	-	-
$SU(3)$	-	-	-	$\surd$	-	-
$\mathbb{R}^8$	-	-	-	-	$\surd$	-
$SU(2)$	-	-	-	-	$\surd$	-
$\{1\}$	-	-	-	-	-	$\surd$

N denotes the number of parallel spinors and stab their stability subgroup in $Spin(9,1)$. $\surd$ denotes the cases that the parallel spinors occur. $-$ denotes the cases that do not occur.

- There are compact K and non-compact stability subgroups $K \ltimes \mathbb{R}^8$
- If $N = 16$, then the spacetime is locally isometric to $\mathbb{R}^{9,1}$
- Some stab(ϵ) are different from those that appear in the Berger list for Riemannian manifolds

Geometry

Gran, Lohrmann, GP

(i). stab(ϵ) compact

- The spacetime admits 1 timelike, and 2 (G_2), 3 ($SU(3)$) and 5 ($SU(2)$) spacelike $\hat{\nabla}$-parallel one-forms.
- The commutator $[X, Y]$ of any two X, Y, $\hat{\nabla}$-parallel vector fields, and so Killing, is also $\hat{\nabla}$-parallel.
- The commutator is determined by H

Two assumptions

- The parallel spinors are Killing
- The $\hat{\nabla}$-parallel vectors constructed from Killing spinor bilinears span a Lie algebra $\mathfrak{h}$ of a group $\mathcal{H}$.

The spacetime is a principal bundle $M = P(\mathcal{H}, B, \pi)$ equipped with a instanton-like connection λ with curvature $\mathcal{F}$.

The metric and H of the background can be written as

$$ds^2 = \eta_{ab}\lambda^a\lambda^b + \pi^* d\tilde{s}^2$$
$$H = \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda \wedge \mathcal{F}^b + \pi^* \tilde{H}$$

The base space B admits an integrable, conformally balanced K-structure, compatible with a connection, $\hat{\tilde{\nabla}}$, with skew-symmetric torsion associated with the pair $(d\tilde{s}^2, \tilde{H})$.

In addition

$$dH = \eta_{ab}\mathcal{F}^a \wedge \mathcal{F}^b + \pi^* d\tilde{H}$$

i.e. part of dH is specified by the first Pontrjagin form of P

G_2

$\mathfrak{h}= \mathfrak{sl}(2,\mathbb{R})$ or $\mathbb{R}\oplus\mathfrak{u}(1)\oplus\mathfrak{u}(1)$

$$\tilde{H} = -\frac{r}{6}(d\varphi, \star\varphi)\varphi + \star d\varphi + \star(\tilde{\theta}_\varphi \wedge \varphi)$$

Ivanov, et al

$$\tilde{\theta}_\varphi = 2d\Phi \ ,$$
$$d\star\varphi = -\tilde{\theta}_\varphi \wedge \star\varphi$$

$r = 0$ if $\mathfrak{h}$ abelian, and $r = 1$ if $\mathfrak{h}$ non-abelian, where

$$\tilde{\theta}_\varphi = \star(\star d\varphi \wedge \varphi)$$

is the Lee form of the G_2-invariant form φ.

In addition, λ , is a $\mathfrak{h}$-valued, $\mathfrak{g}_2 \subset \Lambda^2(\mathbb{R}^7)$ instanton

$$\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$$

$SU(3)$

$\mathfrak{h} = \mathbb{R} \oplus^3 \mathfrak{u}(1)$, $\mathbb{R} \oplus \mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$, $\mathfrak{so}(2) \oplus_s \mathfrak{h}_2(\mathbb{R})$

If $\mathfrak{h}$ abelian, $\mathrm{hol}(\hat{\hat{\tilde{\nabla}}}) \subseteq SU(3)$ and λ an abelian $\mathfrak{s}u(3) \subset \Lambda^2(\mathbb{R}^6)$ Donaldson connection (B Hermitian).

if $\mathfrak{h}$ non-abelian, $\mathrm{hol}(\hat{\hat{\tilde{\nabla}}}) \subseteq U(3)$ and λ is a $\mathfrak{h}$-valued $\mathfrak{u}(3) \subset \Lambda^2(\mathbb{R}^6)$ Donaldson connection

$SU(2)$

$\mathfrak{h} = \mathbb{R} \oplus^5 \mathfrak{u}(1)$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, $\mathfrak{cw}_6$

$\mathrm{hol}(\hat{\hat{\tilde{\nabla}}}) \subseteq SU(2)$ and λ a $\mathfrak{h}$-valued, instanton on B

(ii) $\mathrm{stab}(\epsilon) = K \ltimes \mathbb{R}^8$ non-compact

- The KSE have also been solved
- M admits a single $\hat{\nabla}$-parallel null vector field, and so Killing, with non-vanishing rotation.
- If the rotation vanishes, the spacetime is a pp-wave propagating in a manifold B with skew-symmetric torsion and a K-structure.

Remark: The analysis can also be extended to $N = 2$ common sector backgrounds.

IIB N-BACKGROUNDS

Gran, Gutowski, Roest, GP

- Focus on invariant Killing spinors under the heterotic stability groups
- The IIB KSE are tractable for maximal and half-maximal number of invariant spinors

e.g. there are 8 $SU(3)$-invariant spinors in IIB.

Maximal $SU(3)$-backgrounds are those with 8 Killing spinors

half-Maximal $SU(3)$-backgrounds are those with 4 Killing spinors.

stab(ϵ)	N = 1	N = 2	N = 3	N = 4	N = 6	N = 8	N = 16	N = 32
$Spin(7) \ltimes \mathbb{R}^8$	$\surd$	$\surd$	-	-	-	-	-	-
$SU(4) \ltimes \mathbb{R}^8$	$\surd$	$\surd$		$\surd$	-	-	-	-
G_2	$\surd$	$\odot$		$\surd$	-	-	-	-
$Sp(2) \ltimes \mathbb{R}^8$	-		$\odot$		$\surd$	-	-	-
$SU(2)^2 \ltimes \mathbb{R}^8$	-			$\odot$		$\surd$	-	-
$SU(3)$	-			$\odot$		$\surd$	-	-
$\mathbb{R}^8$	-					$\odot$	$\surd$	-
$SU(2)$	-					$\odot$	$\surd$	-
$\{1\}$	-						$\odot$	$\surd$

Table: $\surd$ solved cases. $\odot$ cases that can be tackled. – do not occur.

- The maximal G-backgrounds have been classified

IIB G-maximal backgrounds

The geometry is characterized by the stability subgroup

- For $K \ltimes \mathbb{R}^8$, M is a pp-wave propagating in a manifold with holonomy K. New solutions were found.
- For compact stability subgroups K, $M = X_n \times Y_{10-n}$, where Y_{10-n} is a manifold with holonomy K and X_n is a Lorentzian symmetric space
  - G_2: $M = \mathbb{R}^{2,1} \times Y_7$, Y_7 G_2-manifold
  - $SU(3)$: $M = AdS_2 \times S^2 \times Y_6$, $CW_4 \times Y_6$, $\mathbb{R}^{3,1} \times Y_6$, Y_6 Calabi-Yau
  - $SU(2)$: $M = AdS_3 \times S^3 \times Y_4$, $CW_6 \times Y_4$, $\mathbb{R}^{5,1} \times Y_4$, Y_4 hyper-Kähler.

Do all N really occur ?

Gran, Gutowski, Roest, GP

Preons are solutions that preserve 31 supersymmetries.

31 spinors span a hyperplane and have a unique normal ν w.r.t. a suitable inner product in the space of IIB spinors.

The gauge symmetry can be used to choose the normal ν as

stab(ν)	spinor ν
$Spin(7) \ltimes \mathbb{R}^8$	$(a+ib)(e_5+e_{12345})$
$SU(4) \ltimes \mathbb{R}^8$	$(a+ib)e_5+(c+id)e_{12345}$
G_2	$a(e_5+e_{12345})+b(e_1+e_{234})$

Choose the Killing spinors orthogonal to ν. Then

$$\mathcal{A}\epsilon_r = 0\ , \quad r = 1, \dots 31$$

implies that

$$P = G = 0$$

The remaining KSE are linear over the complex numbers and so the number of Killing spinors preserved is even. So there are no IIB preons.

- There are no IIA preons

Bandos, Azcarraga, Izquierdo, et al

- Are there any M-preons?

D=11 N-BACKGROUNDS

$D = 11$ case is less complete. There are many cases where stab$(\epsilon) \subset Spin(10,1)$ is non-trivial,

e.g. $Spin(7) \ltimes \mathbb{R}^9$, $SU(5)$, $SU(4)$, $SU(3)$, $SU(3)\times SU(2)$, $SU(2)\times SU(2)$, $SU(2)$ and others.

The KSE have been solved for the following cases

stab(ϵ)	N = 1	N = 2	N = 3	N = 4
$Spin(7) \ltimes \mathbb{R}^9$	$\surd$	$\surd$	-	-
$SU(5)$	$\surd$	$\surd$	-	-
$SU(4)$	-	$\surd$		$\surd$
$G_2 \ltimes \mathbb{R}^9$	-			$\surd$

Gauntlett, Gutowski, Pakis

Gillard, Gran, Roest, GP

Cariglia, Conamhna

Applications

The $N = 2$ $SU(5)$-backgrounds include the most general M-theory compactifications on CY_{10} with fluxes to one-dimension.

The $N = 4$ $SU(4)$-backgrounds are rotating, wrapped, resolved, membranes on CY_8 which are generalizations of the $M2$-brane

Duff, Stelle

SUMMARY

- There is a good understanding of the geometry of supersymmetric heterotic supergravity backgrounds. The geometry of the common sector $N = 2$ backgrounds is understood. The $N \geq 3$ cases are tractable.
- In IIB supergravity the maximally G-supersymmetric backgrounds have been classified. The half-maximal G-backgrounds are tractable. There are no IIB preons.
- In $D = 11$, the $N = 32$ backgrounds have been classified and the geometry of $N = 1$ and a few more $N = 2$ and $N = 4$ backgrounds has been understood.