Some non-perturbative results in globally supersymmetric gauge theories

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~ SUGRA 30 ~

Plan :

- non-perturbative "non-renormalisation" theorems and structure of F-terms
- some exact results in $\mathcal{N}=2$ susy gauge theories
- ~~some exact results in $\mathcal{N}=1$ susy gauge theories~~

$\rightsquigarrow$ strong constraints on effective action

- To avoid any trouble with IR divergences and possible non-holomorphic dependences: must use Wilsonian effective action

$$e^{i S^{W,\lambda}_{\text{eff}}[\chi]} = \underbrace{\int \mathcal{D}\chi'}_{\substack{\text{1PI only} \\ \text{loop momenta} \gtrsim \lambda}} e^{i S[\chi + \chi']}$$

$\Rightarrow$ using $S^{W,\lambda}_{\text{eff}}$ with a UV cutoff λ gives the same results (at scales $\lesssim \lambda$) as the original S.

- add background fields. Start with

$$\mathcal{L}[Y, T, \phi, V] = [\phi^+ e^{-V} \phi]_D + 2\,\mathrm{Re}[Y f(\phi)]_F + \mathrm{Re}\left[\frac{T}{8\pi i} W^{A\alpha} W^A{}_\alpha\right]_F$$

- note: $[W^{A\alpha} W^A{}_\alpha]_F = -\frac{1}{2} F^A_{\mu\nu} F^{A\mu\nu} + \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^A_{\rho\sigma}$

total derivative $\Rightarrow$ irrelevant in perturbation theory

perturbative non-renormalisation theorems are based on symmetries that change S by $\int \varepsilon F F = 0$ in pert. theory.

symmetries : • $U(1)_R$ $\quad \theta \to e^{i\alpha}\theta, \ \bar\theta \to e^{-i\alpha}\bar\theta$

$$\phi \to \phi \ , \ Y \to e^{2i\alpha} Y$$

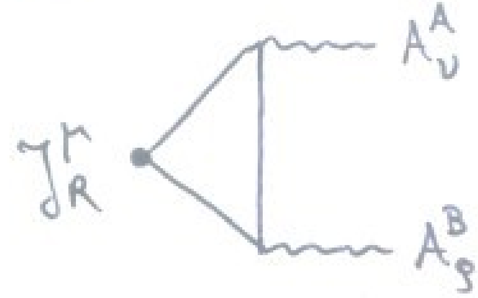
$$V \to V \ , \ W_\beta \to e^{i\alpha} W_\beta$$

$$\phi = \varphi + \theta\psi + \dots \ , \ W = \lambda + \dots$$

$\Rightarrow$ ψ has R-charge -1 and λ has R-charge $+1$

left-chiral

$\Rightarrow$ anomaly



λ are in the adjoint

$$\Rightarrow \quad q_R \, \mathrm{tr}_{adj} \, t_A t_B = + C_1 \, \delta_{AB}$$

ψ are in some "matter" representation $\mathcal{R}$

$$\Rightarrow \quad q_R \, \mathrm{tr}_{\mathcal{R}} \, t_A t_B = - C_2 \, \delta_{AB}$$

$$\frac{\delta \mathcal{L}^w_{eff}}{\delta \alpha} = -\partial_\mu \langle j_R^\mu \rangle = -\frac{1}{32\pi^2}(C_1 - C_2)\,\varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^A_{\rho\sigma}$$

- in perturbation theory $\quad \delta \int \mathcal{L}^w_{eff} = 0$
- in general: $\quad \delta \mathcal{L}^w_{eff} = -2(C_1 - C_2)\,\nu$

$$\nu = \frac{1}{64\pi^2} \int \varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^A_{\rho\sigma}$$

- shift symmetry in T: $T \to T + \xi$, $\xi \in \mathbb{R}$

$$\Rightarrow \quad \delta\mathcal{L} = \mathrm{Re}\left[\frac{\xi}{8\pi i} W^{A\alpha} W^{A}_{\alpha}\right]_F = \frac{\xi}{32\pi} \varepsilon^{\mu\nu\rho\sigma} F^{A}_{\mu\nu} F^{A}_{\rho\sigma}$$

again a symmetry of S_{eff} in perturbation theory but violated non-perturbatively e.g. by instantons.

* combined $U(1)_R / \xi$ -transformation

for $\xi = \frac{c_1 - c_2}{\pi} \alpha$ this is a symmetry of $\mathcal{L}^{w}_{eff}$ in general

(for $c_1 = c_2$ both symmetries are valid separately)

changes under $U(1)_R / \xi$:

$$\begin{array}{lcl} V, \phi & : & 0 \\ W & : & 1 \\ Y & : & 2 \\ e^{2\pi i T} & : & 2(c_1 - c_2) \end{array}$$

$$\Rightarrow \mathcal{L}^{w}_{eff} = \left[\mathcal{K}(\phi, \phi^{+}, V, W, \bar{W}, T, Y, T^{+}, Y^{+})\right]_D + \mathrm{Re}\left[\frac{T}{8\pi i} W^{A\alpha} W^{A}_{\alpha}\right]_F + 2\,\mathrm{Re}\left[G(\phi, W, Y, e^{2\pi i T})\right]_F$$

holomorphic function of total charge 2

- Only positive powers of $e^{2\pi i T}$ can occur in G:

in the end one replaces $T \to \tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}$

$$\Rightarrow \left(e^{2\pi i T}\right)^a \to e^{-8\pi^2 a/g^2 + i a \vartheta}$$

only $a > 0$ correctly gives a suppression for $g \to 0$.

$\Rightarrow$ the $U(1)_R/\xi$ charge of $\left(e^{2\pi i T}\right)^a$ is $2a(C_1 - C_2)$ and has the same sign as $C_1 - C_2$.

- if $C_1 > C_2$: all fields entering G have charge ≥ 0 $\Rightarrow$ a limited number of possibilitie to get charge 2
- if $C_1 = C_2$: still charge ≥ 0 but arbitrary dependence on $e^{2\pi i T}$
- if $C_1 < C_2$: possible compensations between positive and negative charges $\to$ much less constrained.

example: susy $SU(N_c)$ QCD : split the non-chiral quarks (N_c-repres.) into left-handed and right-handed. Describe the right-handed by their charge-conjugated left-handed in $\bar{N}_c$ repres. $\Rightarrow$ two left-chiral multiplets, one in N_c one in $\bar{N}_c$ repres.

Each chiral multiplet comes in N_f copies

$$\Rightarrow \mathrm{tr}_{adj}\, t_A t_B = N_c \delta_{AB} \quad : \quad C_1 = N_c$$

$$\mathrm{tr}_R\, t_A t_B = N_f\, \mathrm{tr}_{N_c} \ldots + N_f\, \mathrm{tr}_{\bar{N}_c} \ldots = 2 N_f \tfrac{1}{2} \delta_{AB} : C_2 = N_f$$

The case $C_1 > C_2$ in detail:

$e^{\frac{2\pi i T}{C_1 - C_2}}$ has $U(1)_R/\xi$-charge $+2$

$$\Rightarrow G = Y g(\phi) + h_{AB}(\phi) W^{A\alpha} W^A_\alpha + e^{\frac{2\pi i T}{C_1 - C_2}} \hat{f}(\phi)$$

charges $2 \quad 0 \quad , \quad 0 \quad 1 \quad 1 \quad , \quad 2 \quad 0$

- let $\langle Y \rangle = y$, $\langle T \rangle = \tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}$

  in $\mathcal{L}$: $y f(\phi) \equiv \sum_k y g_k [\phi]^k$ ← some gauge inv. combination

  $g, y \to 0$ is like taking <u>all</u> tree-level couplings to zero

  Since $g(\phi)$ is independent of y it must be given by the tree-level superpotential:

$$g(\phi) = f(\phi)$$

- The rest of G is indep. of Y $\Rightarrow$ take $Y = 0$.

  $\Rightarrow$ new symmetry of $\mathcal{L}|_{Y=0}$:

  $\phi \to e^{i\delta} \phi$, $\phi^+ \to e^{-i\delta} \phi^+$

  This is again anomalous. However,

we can combine it with the appropriate shift of T, and to obtain the T-independent $h_{AB}(\phi)$ let

$\langle T \rangle \to i\infty \Leftrightarrow g \to 0$ (and $\Upsilon = 0$)

$\Rightarrow$ $h_{AB}(\phi)$ must contain equal numbers of ϕ and ϕ^+ $\to$ is ϕ indep. By gauge invariance:

$$h_{AB} = \underbrace{h_\lambda}_{\text{scale (IR cutoff) dependent}} \delta_{AB}$$

$$\mathcal{L}^{W,\lambda}_{\text{eff}} = [\dots]_D + \mathrm{Re}\left[\left(\frac{\tau}{8\pi i} + h_\lambda\right) W^{A\alpha} W^A_\alpha\right]_F + 2\,\mathrm{Re}\Big[\underset{\substack{\uparrow \\ \text{tree-level} \\ \text{superpotential}}}{f(\phi)} + \underset{\substack{\uparrow \\ \text{non-perturbative} \\ \text{superpotential}}}{e^{\frac{2\pi i \tau}{c_1 - c_2}} \hat{f}(\phi)}\Big]_F$$

$$\tau_{\text{eff}}(\lambda) = 4\pi i \underbrace{\left(\frac{1}{g^2} + 2\,\mathrm{Re}\, h_\lambda\right)}_{\substack{\text{tree-level plus} \\ \text{one-loop}}} + \frac{1}{2\pi}\big(\vartheta \underbrace{- 8\pi\, \mathrm{Im}\, h_\lambda}_{\substack{\text{cannot be a higher-order} \\ \text{correction} \to \text{must vanish}}}\big)$$

$\Rightarrow \vartheta$ is not renormalised

must be $\dfrac{1}{g^2(\lambda)} = \dfrac{1}{g^2} - 2b \log\dfrac{\lambda}{\lambda_0} \equiv -2b\log\dfrac{\lambda}{\Lambda}$

$$\lambda \frac{d}{d\lambda} g(\lambda) = b\, g^3(\lambda), \quad b = -\frac{1}{16\pi^2}(3c_1 - c_2)$$

$$e^{\frac{2\pi i \tau}{c_1 - c_2}} \hat{f}(\phi) \equiv e^{\frac{2\pi i \tau(\lambda)}{c_1 - c_2}} \hat{f}_\lambda(\phi) = e^{\frac{i\vartheta}{N_c - N_f}} \left(\frac{\Lambda}{\lambda}\right)^{\frac{3c_1 - c_2}{c_1 - c_2}} \hat{f}_\lambda(\phi)$$

$$\Rightarrow \mathcal{L}^{W,\lambda}_{\text{eff}} = [\dots]_D + \mathrm{Re}\left[\frac{\tau_{\text{eff}}(\lambda)}{8\pi i} W^{A\alpha} W^A_\alpha\right]_F + 2\,\mathrm{Re}\left[f(\phi) + \left(\frac{\Lambda}{\lambda}\right)^{\frac{3c_1 - c_2}{c_1 - c_2}} \hat{f}_\lambda(\phi)\right]$$

- Remains to determine $\hat{f}_\lambda(\phi)$ from any other global flavour symmetries:

example of <u>$SU(N_c)$ super QCD</u>

chiral multiplets: Q_{ai} , $\overline{Q}_{aj}$ $\leftarrow$ $\overline{N_c}$-repres.

N_c-repres, $SU(N_f)$

$C_1 = N_c$, $C_2 = N_f$

Flavour-symmetry $SU(N_f) \times SU(N_f)$ separately on Q and $\overline{Q}$.

Only flavour and gauge invariant is

$$\varepsilon^{i_1 \dots i_{N_f}} \varepsilon^{j_1 \dots j_{N_f}} Q_{a_1 i_1} \dots Q_{a_{N_f} i_{N_f}} \overline{Q}_{a_1 j_1} \dots \overline{Q}_{a_{N_f} j_{N_f}}$$

$$\sim \mathrm{Det}_{i,j} \Big(\sum_a Q_{ai} \overline{Q}_{aj} \Big) \equiv D$$

and $\hat{f}_\lambda(Q, \overline{Q})$ must be $\hat{f}_\lambda(D)$

The $SU(N_f)$ was potentially anomalous, but

$\partial_\mu j^\mu_{\text{flavour}}$  $\sim (\mathrm{tr}\, t^{SU(N_f)}) = 0$

$U(1) \times U(1)$ symmetry: $Q \to e^{i\varphi} Q$, $\overline{Q} \to e^{i\tilde{\varphi}} \overline{Q}$

$\Rightarrow D \to e^{i N_f (\varphi + \tilde{\varphi})} D$

but this is again anomalous and $\mathrm{tr}\, t^{U(1)} \neq 0$:

$$\delta \mathcal{L}^{w}_{eff} = -\frac{1}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma} F^{A}_{\mu\nu} F^{A}_{\rho\sigma}\left(N_f \tfrac{1}{2}\varphi + N_f \tfrac{1}{2}\tilde{\varphi}\right)$$

which can again be cancelled by $T \to T + \frac{N_f}{2\pi}(\varphi+\tilde{\varphi})$

$$\Rightarrow e^{\frac{2\pi i T}{N_c-N_f}}\,\hat{f}_\lambda(D) \to e^{i\frac{N_f}{N_c-N_f}(\varphi+\tilde{\varphi})}\, e^{\frac{2\pi i T}{N_c-N_f}}\,\hat{f}_\lambda\left(e^{iN_f(\varphi+\tilde{\varphi})}D\right)$$

invariant if $\hat{f}_\lambda(D) = c_\lambda D^{-\frac{1}{N_c-N_f}}$.

Now count mass dimensions:

$$[\hat{f}] = 3,\quad [D] = 2N_f \Rightarrow [c_\lambda] = 3 + \frac{2N_f}{N_c-N_f} = \frac{3N_c-N_f}{N_c-N_f}$$

$$\Rightarrow c_\lambda = c\,\lambda^{\frac{3N_c-N_f}{N_c-N_f}}$$

The complete non-perturbative superpotential is:

$$f_{tree}(Q,\bar{Q}) + e^{\frac{i\vartheta}{N_c-N_f}}\left(\frac{\Lambda}{\lambda}\right)^{\frac{3N_c-N_f}{N_c-N_f}} c_\lambda D^{-\frac{1}{N_c-N_f}} =$$

$$W_{n.p.}(Q,\bar{Q}) = f_{tree}(Q,\bar{Q}) + c\,e^{\frac{i\vartheta}{N_c-N_f}}\Lambda^{\frac{3N_c-N_f}{N_c-N_f}}\left(\mathrm{Det}\,Q_{ai}\bar{Q}_{aj}\right)^{-\frac{1}{N_c-N_f}}$$

all non-perturbative effects
are contained in a single constant
(dimensionless) number

$\to$ needs to do one computation

The case $C_1 = C_2$

examples:
- pure $\mathcal{N}=2$ super YM
- any $\mathcal{N}=1$ sYM theory with an adjoint chiral multiplet with a tree-level superpotential

$C_1 = C_2 \Rightarrow$ the $U(1)_R$ is anomaly-free but no shift symmetry for T

$\Rightarrow$ all terms in G must have $U(1)_R$-charge 2:

$\phi : 0 \quad , \; Y : 2 \quad , \; e^{2\pi i T} : 0 \, ,$

$W : 1$

$$\Rightarrow \quad G = \underset{\substack{\uparrow \\ 2}}{Y} \; g(\underset{\substack{\uparrow \\ 0}}{\phi}, \underset{\substack{\uparrow \\ 0}}{e^{2\pi i T}}) + h_{AB}(\underset{\substack{\uparrow \\ 0}}{\phi}, \underset{\substack{\uparrow \\ 0}}{e^{2\pi i T}}) \underset{\substack{\uparrow \\ 1}}{W^{A\alpha}} \underset{\substack{\uparrow \\ 1}}{W^{B}_{\alpha}}$$

charges: 2, 0, 0, 0, 0, 1, 1

Scaling $Y \Rightarrow g(\phi, e^{2\pi i T})$ depends linearly on all parameters that entered linearly in the tree-level superpotential $f(\phi)$ ($\rightarrow$ linearity principle)

$\Rightarrow$ If the tree-level superpotential is zero then no superpotential is generated!

(corresponds to $Y \rightarrow 0$) This is the case of $\mathcal{N}=2$.

- anomalous symmetries are more powerful:

$$\phi \to e^{i\delta}\phi \qquad \text{(anomalous)}$$

let in $f(\phi)$: $\sum_k g_k \phi^k \to \sum_k g_k Y_k \phi^k$

$$Y_k \to e^{-ik\delta} Y_k$$

and to cancel the anomaly $\sim C_2$

$$T \to T + \frac{C_2}{\pi}\delta$$

$U(1)$ - charges: $\phi : 1$, $Y_k : -k$, $e^{2\pi i a T} : 2C_2 a$

$a \geq 0$!

→ terms in $h_{AB}(\phi, e^{2\pi i T})$:

$$h\,\delta_{AB} + \underbrace{\sum_{\nu\geq 1}^{\infty} h_{AB}^{\nu} \left(\frac{e^{2\pi i T}}{\phi^{2C_2}}\right)^{\nu}}_{\text{infinite instanton series}}$$

one-loop contribution to $\tau_{eff}(\lambda)$ ↴

$+h_1 \delta_{AB} \log\left(e^{2\pi i T}/\phi^{2C_2}\right)$

→ terms in $g(\phi, e^{2\pi i T})$:

$$g(\phi, e^{2\pi i T}) = \underbrace{f(\phi)}_{\text{tree-level superpotential}} + \underbrace{\sum_k \underbrace{g_k}_{\text{couplings of } f(\phi)} \phi^k \sum_{\nu \geq 1}^{\infty} \underbrace{g_{k,\nu}}_{\text{constants}} \left(\frac{e^{2\pi i T}}{\phi^{2C_2}}\right)^{\nu}}_{\text{infinite instanton series}}$$

1.) Non-perturbative "non-renormalisation" theorems and structure of F-terms

Use $\mathcal{N}=1$ superfields

$$\Phi = \varphi + \sqrt{2}\theta\psi - \theta\theta f \qquad \text{(chiral superfield)}$$

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)F_{\mu\nu} + \dots$$

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^{V}\right)$$

Susy Lagrangian:

$$\mathcal{L} = \left[\Phi^{+}e^{-V}\Phi\right]_D + 2\,\mathrm{Re}\left[f(\Phi)\right]_F + \mathrm{Re}\left[\frac{\tau}{8\pi i}W^{A\alpha}W^{A}_{\alpha}\right]_F$$

tree-level superpotential
= holomorphic fct of Φ

$$\tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}$$

Seiberg's idea: replace coupling constants by chiral background fields with non-trivial transformation properties

$\rightsquigarrow$
- obtain larger symmetry groups
- the F-terms of the effective action must be holomorphic in these background fields
- use extra input from various limits

For $SU(2)$: Seiberg-Witten solution

There is now a single abelian massless $\mathcal{N}=2$ multiplet (W_α, ϕ) and $\mathcal{L}^W_{eff}$ depends on the modulus $u = \langle tr\, \varphi^2 \rangle$.

We still do not have much control over the D-term but restricting to a

low energy effective action

keeping only the leading 2-derivative terms:

$$\mathcal{L}^{W,\,low\,en}_{eff} = \left[K(\phi,\phi^\dagger)\right]_D + Re\left[T(\phi)W^\alpha W_\alpha\right]_F$$

where any τ-dependence is hidden in the coefficients of $T(\phi)$.

$\mathcal{N}=2$ imposes: $K(a,a^*) = Im\left(a^* \frac{h(a)}{4\pi}\right)$

$$T(a) = \frac{1}{4\pi i}\, h'(a)$$

or with $h(a) = \mathcal{F}'(a)$:

$$\mathcal{L}^{W,\,low\,en}_{eff} = \frac{1}{4\pi} Im\left(\left[\phi^\dagger \mathcal{F}'(\phi)\right]_D + \left[\mathcal{F}''(\phi)W^\alpha W_\alpha\right]_F\right)$$

The Kähler potential of the D-term is determined by the same holomorphic function $\mathcal{F}$ as is the F-term!

We know that $\frac{1}{4\pi i}\mathcal{F}''(\phi) \sim h_{AB}(\phi, e^{2\pi i\tau}) + \frac{\tau}{4\pi i}\delta_{AB}$

$\Rightarrow$ $\mathcal{F}''(\phi)$ = tree-level + 1-loop + infinite instanton series

$$\text{for } SU(2): c_1 = C_2 = 2 \,, \quad e^{2\pi i \tau_\lambda} = \left(\frac{\Lambda}{\lambda}\right)^{2c_1} = \left(\frac{\Lambda}{\lambda}\right)^4$$

$$\mathcal{F}''(\phi) = \underbrace{\tau_\lambda + 2h_2 \log\frac{\Lambda^2}{\phi^2}}_{\overset{!}{=} \frac{i}{\pi}\left(\log\frac{\phi^2}{\Lambda^2}+3\right)} + \sum_{n=1}^{\infty} c_n \left(\frac{\Lambda}{\phi}\right)^{4n}$$

from one-loop computation.

We want to interpret $\mathcal{F}''(a)$ as the effective $\tau_{eff} = \frac{4\pi i}{g^2_{eff}} + \frac{\vartheta_{eff}}{2\pi}$

$$\Rightarrow \quad \operatorname{Im}\mathcal{F}''(a) \geq 0$$

But $\operatorname{Im}\mathcal{F}''$ is a harmonic function and hence cannot have a minimum unless it is constant

$\Rightarrow$ $\mathcal{F}$ cannot be globally defined on the moduli space $\rightarrow$ use different patches with transition functions realised by electric-magnetic duality

$$\begin{pmatrix} a \\ a_D \equiv \mathcal{F}'(a) \end{pmatrix} \rightarrow \begin{pmatrix} a_D \\ -a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ a_D \end{pmatrix}$$

$$\mathcal{F}_D'(a_D) = -a \quad \Rightarrow \quad \mathcal{F}_D''(a_D) = -\frac{1}{\mathcal{F}''(a)}$$

$\Rightarrow$ the coupling gets inverted

$\mathcal{F}'' \to \mathcal{F}'' + l$ is $\tau \to \tau + l \Leftrightarrow \vartheta \to \vartheta + 2\pi l$

$\Rightarrow$ full duality group also $a_D \to a_D + l a$, $l \in \mathbb{Z}$

$\Rightarrow$ generate $Sl(2,\mathbb{Z})$ or $Sp(2,\mathbb{Z})$

- BPS states:

$$m = \sqrt{2}\, | n_m a_D - n_e a |$$

$\uparrow$ $Sp(2,\mathbb{Z})$ invariant.

- an anomalous $U(1)_R$ again

$\theta \to e^{i\alpha}\theta$, $W \to e^{i\alpha} W$, $\phi \to e^{2i\alpha}\phi$ is a symmetry of the $\mathcal{N}=2$ Lagrangian. $\Leftrightarrow \lambda \to e^{i\alpha}\lambda, \psi \to e^{i\alpha}\psi$

anomaly: $\delta \int \mathcal{L}_{eff} = -\frac{\alpha}{32\pi^2}(C_1 + C_1)\,\varepsilon^{\cdots} F_{..}F_{..} = -8\alpha\nu$

$$\Rightarrow e^{i\delta\int\mathcal{L}_{eff}} = e^{-8i\nu\alpha} \overset{!}{=} 1 \quad \text{if } \alpha = \frac{2\pi}{8} \cdot k$$

$\Rightarrow \mathbb{Z}_8$-symmetry is left unbroken

$\mathbb{Z}_8$ must be a symmetry of the effective action (up to duality transformations)

Under $\mathbb{Z}_8$: $\quad u = \langle \mathrm{tr}\,\varphi^2 \rangle \to e^{4i\alpha} u \equiv (-)^k u$

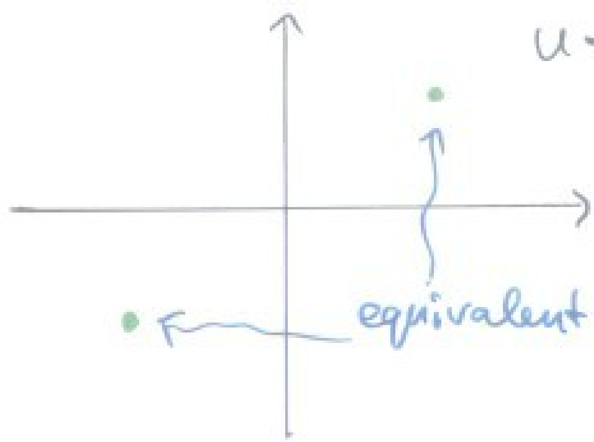
$k = 0, 1, \dots 7$

$\Rightarrow$ at a given point in moduli space a $\mathbb{Z}_4$ is preserved

$\Rightarrow$ $\mathbb{Z}_8 / \mathbb{Z}_4 = \mathbb{Z}_2$ acts as a symmetry on moduli space

$\Rightarrow$ the theory at u is equivalent to the theory at $-u$

$\Rightarrow$ physical "observables" like the BPS mass spectrum must be invariant under $u \leftrightarrow -u$



- determining $\mathcal{F}(u)$ or $\begin{pmatrix} a(u) \\ a_D(u) \end{pmatrix}$:

. one-loop result (valid at scale $u \to \infty$ by asymptotic freedom):

$$a \sim \sqrt{2u}$$

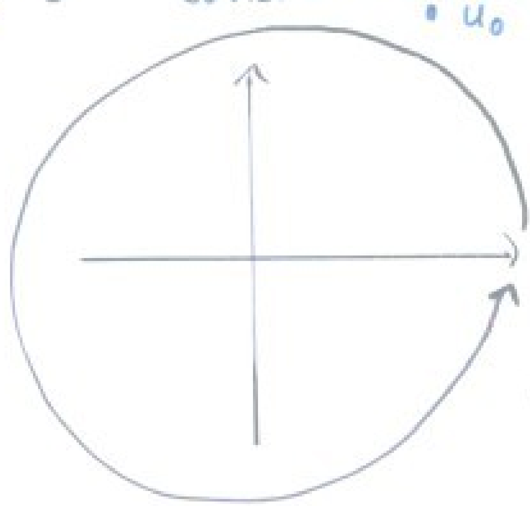
$$a_D \sim \frac{i}{\pi} a \left(\log \frac{a^2}{\Lambda^2} + 1 \right) \sim \frac{i}{\pi} \sqrt{2u} \left(\log \frac{2u}{\Lambda^2} + 1 \right)$$

$u = \infty$ is a branch point.

Monodromy around a singular point u_0:

$u \to u(t) = u_0 + \varepsilon e^{it}$ and take t from 0 to 2π

for $u_0 = \infty$ consider $u' = \frac{1}{u_0}$ instead:



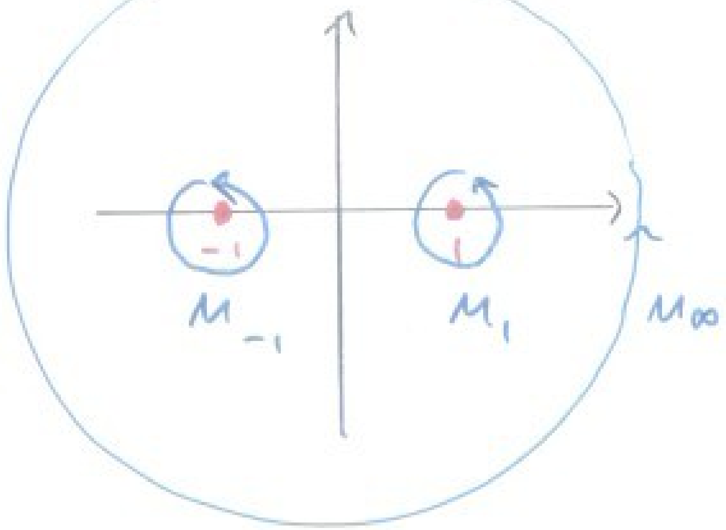
$$\begin{pmatrix} a(u) \\ a_D(u) \end{pmatrix} \to \begin{pmatrix} -a(u) \\ -a_D(u) + 2a(u) \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}}_{M_\infty} \begin{pmatrix} a(u) \\ a_D(u) \end{pmatrix}$$

- Analytic structure: if only 2 branch points
  $\to$ log-type function
  $\Rightarrow$ $\operatorname{Im} \mathcal{F}''$ not positive definite

$\Rightarrow$ need (at least) 3 singularities

by the $\mathbb{Z}_2$-symmetry: $\infty, u_0, -u_0$

by redefining Λ: $\infty, 1, -1$



consistency $M_\infty = M_1 M_{-1}$

- singularity at $u=1$ due to some heavy particle becoming massless
  (SW: cannot be any of the perturbative states)

  $\rightarrow$ dyon $(n_m = 1, n_e)$

  $\rightarrow$ take $n_e = 0$ (it doesn't matter in the end)

  $\rightarrow$ magnetic monopole $\rightarrow$ use dual description
  massless electron in $\mathcal{N}=2$ super QED

  $\rightarrow$ known β-fct for dual coupling

$$\mu \frac{d}{d\mu} g_D = \frac{1}{8\pi^2} g_D^3 \quad \Rightarrow \quad a_D \frac{d}{da_D} \tau_D = -\frac{i}{\pi}$$

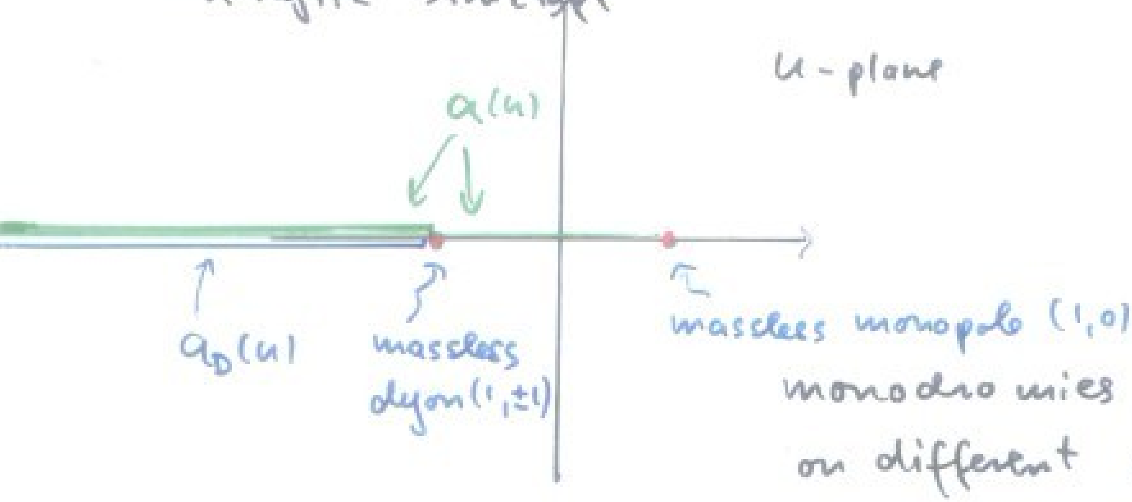
$$\Rightarrow \tau_D = -\frac{i}{\pi} \log a_D \quad . \quad \tau_D = \frac{d(-a)}{da_D}$$

$\Rightarrow$ close to $u_0 \equiv 1$: $a \approx a_0 + \frac{i}{\pi} a_D \log a_D$

from BPS mass-formula $m_{monopole} = \sqrt{2} |n_m a_D|$

$\Rightarrow a_D(1) = 0 \quad \Rightarrow \quad a_D \approx c_0 (u-1)$

analytic structure



monodromies relate a, a_D on different sheets.

$\mathbb{Z}_2$-symmetry : How $\begin{pmatrix} a_D \\ a \end{pmatrix}(u)$ and $\begin{pmatrix} a_D \\ a \end{pmatrix}(-u)$ are related?

$$\begin{pmatrix} a_D(-u) \\ a(-u) \end{pmatrix} = e^{i\omega} \underbrace{G}_{\text{Sp}(2,\mathbb{Z})\text{-matrix}} \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}$$

(depends only on sign of $\text{Im}\, u$)

BPS mass-formula is $Sp(2,\mathbb{Z})$ invariant :

if $\begin{pmatrix} n_m' \\ n_e' \end{pmatrix} = G \begin{pmatrix} n_m \\ n_e \end{pmatrix}$ then

$$|n_m' a_D(-u) - n_e' a(-u)| = |n_m a_D(u) - n_e a(u)|$$

$\Rightarrow$ (n_m', n_e') at $-u$ has the same mass as (n_m, n_e) at u

But physical properties must be the same at u and $-u$ $\Rightarrow$ the set of BPS states at u and $-u$ must be the same

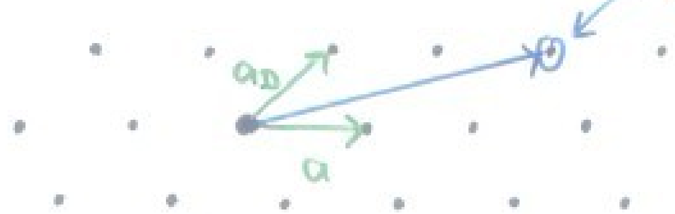
=> The set of allowed (stable) BPS-states must be invariant under G.

Decay of BPS states:

if $n_e = k n_e'$, $n_m = k n_m'$

$$(n_m, n_e) \to k \times (n_m', n_e')$$

stable if n_e and n_m relatively prime.

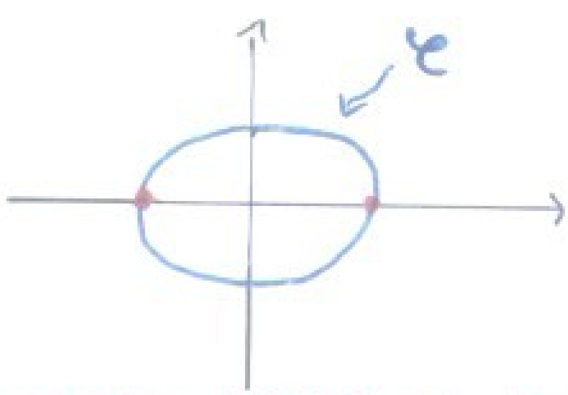


mass = length

=> no way to decay preserving el. & magn. charges

changes if $\vec{a}_D \parallel \vec{a}$, i.e. $\frac{a_D}{a} \in \mathbb{R}$

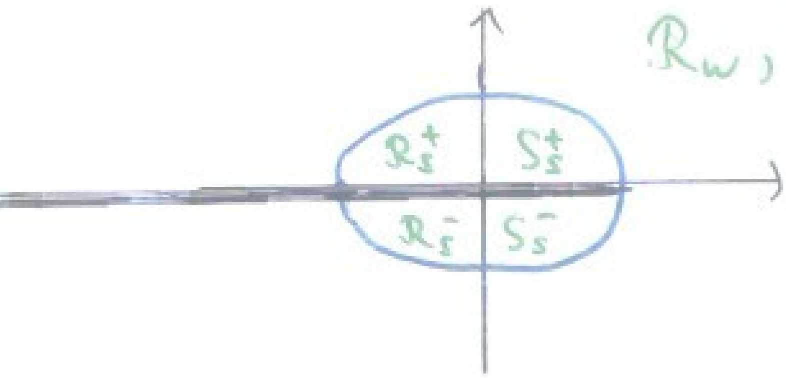
defines a curve on moduli space



massless (BPS) states can only occur on $\mathcal{C}$:

$$n_e a - n_m a_D = 0 \quad \Rightarrow \quad \frac{a_D}{a} = \frac{n_e}{n_m} \in \mathbb{R}$$

(if $n_m = 0$: $n_e a = 0 \Rightarrow n_e = 0$: this is the "photon" described by our $U(1)$ multiplet in $\mathcal{L}_{eff}$)



R_w , S_w

Single weak coupling region R_w with a unique weak coupling spectrum S_w

Strong coupling region R_s^+, R_s^- with different descriptions (related by monodromy M_1) due to the cut. Spectra S_s^+ and S_s^-

$$S_s^+ = M_1 (S_s^-) \Leftrightarrow S_s^- = M_1^{-1} S_s^+$$

example : the massless dyon $(1,-1)$ is in S_s^+

$$M_1^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ +1 \end{pmatrix}$$

the <u>same</u> dyon is described as $(1,1)$ in S_s^-

$\mathbb{Z}_2$-invariance : the weak and strong-coupling spectra must be invariant under the $\mathbb{Z}_2$-transf. G :

$$S_w = G_{w,\pm} S_w$$

upper or lower half plane

$$S_s^{\pm} = G_{s,\pm} S_s^{\pm} \quad , \quad G_{s,\pm} = M_1^{\pm 1} G_{w,\pm}$$

additional input: • on $\mathcal{C}$: $\frac{a_D}{a}$ goes from -1 to 1

• a state (n_m, n_e) with $-1 < \frac{n_e}{n_m} < 1$ cannot exist: it would be massless at the point u where $-1 < \frac{a_D}{a} = \frac{n_e}{n_m} < 1$

this is different from $u = -1, +1$,

-> would imply <u>extra</u> massless states.

$\Rightarrow$ (with Ferrari)

$$S_W = \left\{ \pm(0,1),\ \pm(1, n_e) \right\}$$

$$S_S^+ = \left\{ \pm(1,0),\ \pm(1,-1) \right\}$$

$$\Leftrightarrow S_S^- = \left\{ \pm(1,0),\ \pm(1,1) \right\}$$

S_W contains the "W-bosons" and the dyons with $|n_m| = 1$

S_S contains just the 2 states that can become massless: the monopole and $(1,\pm 1)$ (and their antiparticles).

$\leadsto$ the strong-coupling spectrum can be very different from "naive" expectations: does not even contain the "elementary" states of the original action