An invitation to Anomalies

Consistency conditions on Supergravity theories in diverse dimensions

Paris, October 16th 2006

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Outline

1. Symmetries in Physics. Different realizations
2. The simplest possible example with most of the ingredients
3. Chirality, mass and regulators. Heuristics
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5. A lightning review of index theory and characteristic polynomials
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Symmetries in Physics

Wigner-Weyl $[Q, H] = 0 \qquad Q\,|0\rangle = 0$

Nambu-Goldstone $[Q, H] = 0 \qquad Q\,|0\rangle \neq 0$

Anomalous symmetries

They hold at the classical level, but not quantum mechanically

Anomalies in global currents. Often they are welcome:

Neutral pion decay

Scale invariance: beta function in field theory

Anomalies in gauge currents: They are fatal

Gauge (and diffeomorphism) symmetries are redundancies
in our description of the physical degrees of freedom

In four dimensions the photon and the graviton have only two degrees of freedom

Gauss's law is required to set the number of physical degrees of freedom straight

In the gravitational case we need the diffeomorphism and the hamiltonian constraints

With local currents, conservation is not a luxury, it is needed for the physical consistency of the theory

Unitarity and/or renormalizability of the theory break down. You are not describing the physics you want to describe if anomalies appear

What we verify with anomaly computations is that the theory contains the required degrees of freedom, which is tantamount to preserving some basic symmetries.

Computing an anomaly

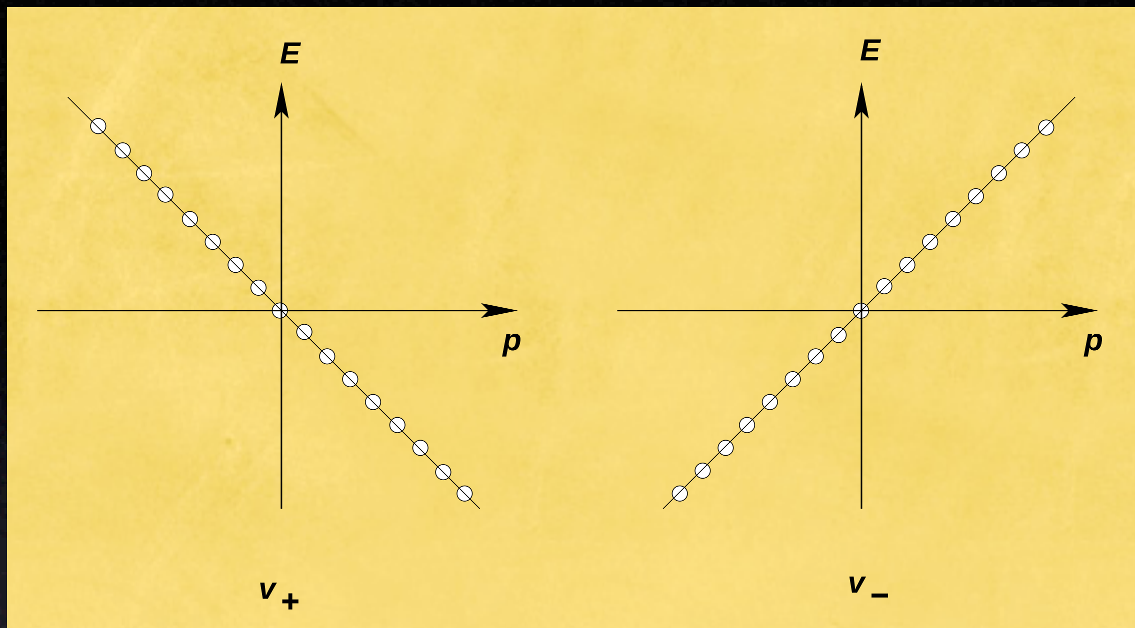
This is the only example where you will see the explicit details. In other cases details will be scarce, but the physics is essentially the same. So, pay attention!

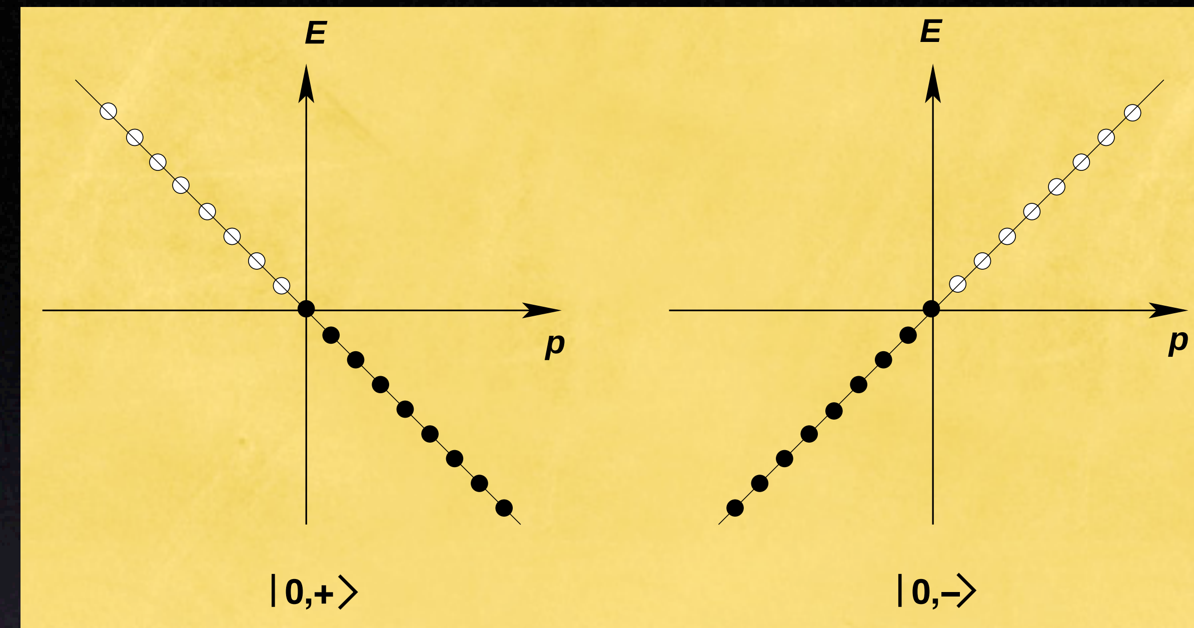
We consider a two-dimensional problem to start with

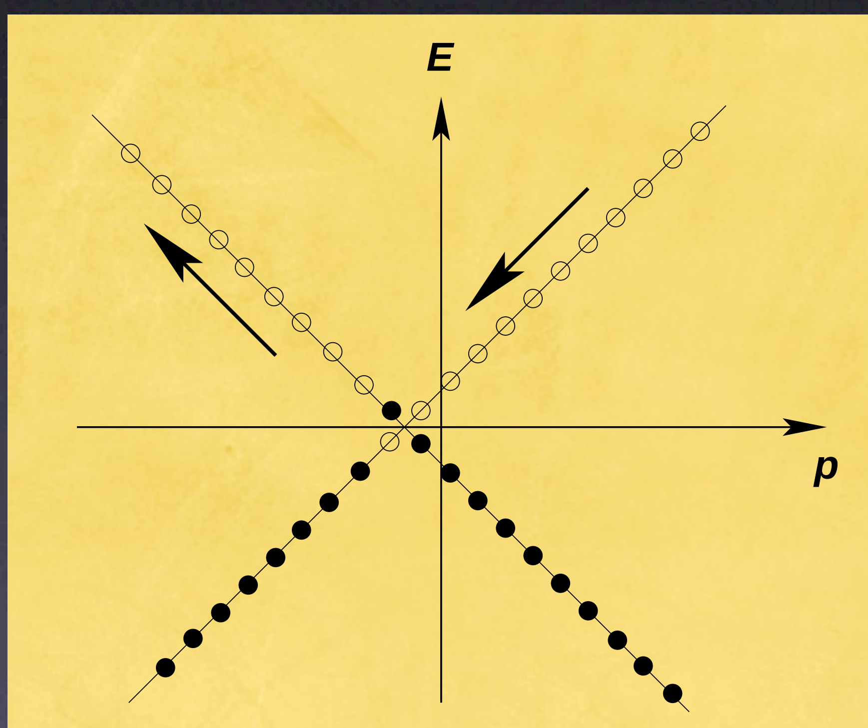
$$\mathcal{L} = \overline{\psi} i \gamma^\mu (\partial_\mu - eA_\mu)\psi \qquad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \qquad \gamma^0 = \sigma^1 \qquad \gamma^1 = i\,\sigma^2 \qquad \gamma_5 = -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have a left and a right moving fermion and we the action is formally invariant under independent change of the phase (if we compensate with the appropriate gauge transformation) We have a left moving current and a right moving current. Both of them are anomalous as soon as a gauge field is turned on. If these currents are associated to true gauge symmetries then the theory is clearly inconsistent. The procedures we follow in the simple example are generalized to higher dimensions later on

$$(\partial_0 - \partial_1)u_+ = 0, \qquad (\partial_0 + \partial_1)u_- = 0$$







First we compute the spectrum

Then we construct the second quantized vacuum

It is like defining a determinant, they represent fermionic degrees of freedom

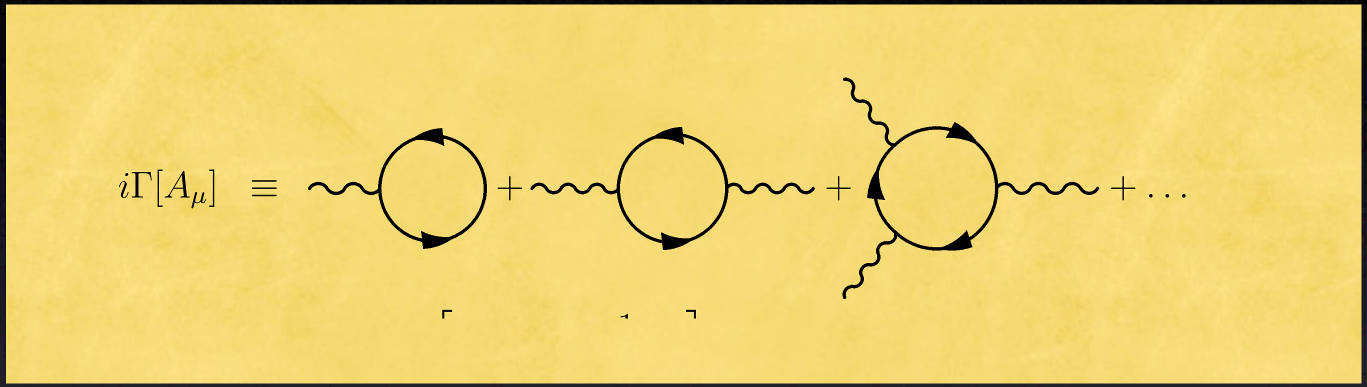
When the gauge field is turned on, the left and right handed currents are not conserved, but the vector current is. The anomaly is related to the chiral structure of the theory. Vector like currents represent no problem

These lessons are easy to generalize, but we need more sophisticated technology

Chirality, mass and regulators. Heuristics

We are evaluating the effective action for the fermions in the presence of external fields

$$\exp(-\Gamma_{eff}[A]) = \text{“Det}D(A)\text{”}$$



$$\Gamma_+[A] = \text{parity invariant} = Re\Gamma_{eff}[A]$$

$$\Gamma_-[A] = \text{parity violating} = Im\,\Gamma_{eff}[A]$$

In the first case we can always find a mass term and regulate the theory preserving gauge invariance

When parity invariant does not hold, the effective action is complex, in general there is no a priori regulartor, and anomalies may appear

Gauge invariance and current conservation are intimately connected

$$\exp(-\Gamma_{eff}[A]) = \text{``Det} D(A)\text{''} = \int d\psi d\overline{\psi} \exp(\int \overline{\psi}\, D(A)\, \psi)$$

$$\Gamma[A_\mu - D_\mu \epsilon] = \int \mathrm{Tr}\, \epsilon\, D_\mu \frac{\delta\, \Gamma}{\delta\, A_\mu} \sim \int D_\mu j^\mu$$

Similarly, with diffeomorphisms we test energy-momentum conservation $\frac{\delta\, \Gamma}{\delta\, g^{\mu\nu}} \sim T_{\mu\nu}$ $\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$

Gauge anomalies can appear in any even number of dimensions.
This depends of course on the fermion representation of the gauge group

In dimension 2n, the first anomalous graph is an n+1 -gon $\epsilon_{\mu_1 \mu_2 \ldots \mu_{2n}}$

In one of the vertices we test current conservation and in the other we have the polarization tensors, there are n independent momenta and n independent polarization vectors

Pure gravity anomalies only occur in 4n+2 dimensions

$$\Gamma = \gamma^0 \gamma^1 \ldots \gamma^{d-1}$$

$$[\mathcal{C}, \Gamma] = 0 \qquad \text{in } d = 4n+2$$

Γ Is the equivalent of the chirality operator in d=4 γ_5

CPT transforms particles of given helicity into particles of the same helicity
We can have chirally asymmetric gravitational couplings

$$\{\mathcal{C}, \Gamma\} = 0 \qquad \text{in } d = 4n$$

Precisely the opposite of the previous case

Feynman graphs vs Index Theory

The anomaly computes the change of the fermion determinant under infinitesimal gauge transformations.

The result of testing current conservation will be a polynomial in the external gauge and gravitational fields

We can use Feynman graphs, or use a deep result in mathematics: THE ATIYAH-SINGER INDEX THEOREM

The AS index theorem also applies in more general settings, and it is necessary for global anomalies, however one should not track of the fact that the result obtain is purely perturbative

$$\mathrm{det} D_L(A,e) \qquad \mathrm{det} D_R(A,e) \qquad D = \begin{pmatrix} 0 & D_L \\ D_R & 0 \end{pmatrix}$$

$$\mathrm{Index} D = \dim Ker D_L - \dim Ker D_R$$

$$= \int_M ch(F) \hat{A}(TM)$$

We now explain the meaning of the symbols under the integral sign

$$F = dA + A^2 = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$$

F is an antihermitian matrix of two-forms

$$\frac{\Omega}{2\pi} = d\omega + \omega^2 = \frac{1}{4\pi} R_{ab\mu\nu}\, dx^\mu dx^\nu$$

We have an antisymmetric matrix of two-forms that can be skew-diagonalized formally

$$\frac{\Omega}{2\pi} \sim \begin{pmatrix} 0 & x_1 & 0 & \dots \\ -x_1 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \end{pmatrix}$$

In 2n dimensions there are n skew eigenvalue two-forms

$$ch(F) = \mathrm{Tr}\, e^{iF/2\pi} \qquad \hat{A}(TM) = \prod_i \frac{x_i/2}{\sinh x_i/2}$$

Chern character, and A-roof genus respectively. The second is a symmetric polynomial in the eigenvalues x_i, hence it can be written always in terms of products of traces of the curvature two-forms

$$\hat{A}(M) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\,\mathrm{Tr}\,R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}(\mathrm{Tr}\,R^2)^2 + \frac{1}{360}\,\mathrm{Tr}\,R^4\right]$$
$$+ \frac{1}{(4\pi)^6}\left[\frac{1}{10368}(\mathrm{Tr}\,R^2)^3 + \frac{1}{4320}\mathrm{Tr}\,R^2\,\mathrm{Tr}\,R^4 + \frac{1}{5670}\,\mathrm{Tr}\,R^6\right]$$
$$+ \frac{1}{(4\pi)^8}\left[\frac{1}{497664}(\mathrm{Tr}\,R^2)^4 + \frac{1}{103680}(\mathrm{Tr}\,R^2)^2\,\mathrm{Tr}\,R^4\right.$$
$$\left. + \frac{1}{68040}\,\mathrm{Tr}\,R^2\,\mathrm{Tr}\,R^6 + \frac{1}{259200}(\mathrm{Tr}\,R^4)^2 + \frac{1}{75600}\,\mathrm{Tr}\,R^8\right] + \ldots$$

We expand formally in terms of forms of different degree both the gravity part and ...

$$\mathrm{ch}(F) = r + \frac{i}{2\pi}\mathrm{Tr}\,F + \frac{i^2}{2(2\pi)^2}\mathrm{Tr}\,F^2 + \ldots + \frac{i^n}{n!(2\pi)^n}\,\mathrm{Tr}\,F^n + \ldots$$

...the gauge part

$$\mathrm{ind}\,D_+ = \frac{1}{(2\pi)^2}\int_M\left(\frac{i^2}{2}\,\mathrm{Tr}\,F^2 + \frac{r}{48}\,\mathrm{Tr}\,R^2\right) \qquad d = 4$$
$$= \frac{1}{(2\pi)^4}\int_M\left(\frac{i^4}{24}\,\mathrm{Tr}\,F^4 + \frac{i^2}{96}\mathrm{Tr}\,F^2\,\mathrm{Tr}\,R^2 + \frac{r}{4608}(\mathrm{Tr}\,R^2)^2 + \frac{r}{5760}\mathrm{Tr}\,R^4\right) \qquad d = 8$$

Multiply them together, and select the 2n-th form

The characteristic polynomials have important properties:

They are closed, as a consequence of the Bianchi identities

$$F = dA + A^2 \qquad dF + AF - FA = 0 = DF$$

$$d\mathrm{Tr}F^2 = 2\,\mathrm{Tr}FdF = 2\,\mathrm{Tr}FDF - 2\,\mathrm{Tr}[A,F] = 0$$

The difference between two characteristic polynomials computed with two continuouly connected connections differ by a total differential: A Chern-Simons form

$$P_m(F_1) - P_m(F_0) = dQ^0_{2m-1}(A,F)$$

$$A(t) \qquad A(0) = A_0 \qquad A(1) = A_1$$

$$Q^0_3 = \mathrm{Tr}(AdA + \frac{2}{3}A^3)$$

Basic result 1 :

$$\boxed{\delta Q^0_{2m-1} = dQ^1_{2m-2}}$$

Stora-Zumino descent equations

Basic result II :

For each chiral field involved in d=2n dimensions, we determine its Index polynomial, extract the form of degree 2n+2, and then apply the Stora-Zumino descent procedure: This is the ANOMALY

Pure gauge in d=2n :

$$I_{1/2}(F) \sim \pm \mathrm{Tr} F^{n+1} \qquad \pm \, \mathrm{Str} T^{a_1} T^{a_2} \dots T^{a_{n+1}}$$

$SU(3)^3$	$SU(2)^3$	$U(1)^3$
$SU(3)^2\, SU(2)$	$SU(2)\, U(1)$	
$SU(3)^2\, U(1)$	$SU(2)\, U(1)^2$	
$SU(3)\, SU(2)^2$		
$SU(3)\, SU(2)\, U(1)$		
$SU(3)\, U(1)^2$		

Possible anomalies in the Standard Model

In supergravity theories we have to identify the fields that will lead to anomalies, compute the corresponding anomaly polynomials, and verify whether they cancel by themselves or with the help of counterterms

Chiral fermions	Chiral gravitini	Self-dual RR fields
ψ	ψ_μ	$\psi_{ab} \sim F_{\mu_1 \mu_2 \dots \mu_n} \quad *F = \pm F$
$I_{1/2} = \prod_i \frac{x_i/2}{\sinh x_i/2}$	$I_{3/2} = \prod_i \frac{x_i/2}{\sinh x_i/2}(-1 + \sum_j 2\cosh x_j)$	$I_A = -\frac{1}{8} \prod_i \frac{x_i}{\tanh x_i}$

If there are gauge quantum numbers, we have to multiply each of these polynomials by the corresponding Chern characters

We are ready to compute some interesting examples

Type IIB d=10

$$(-I_{1/2} + I_{3/2} + I_A)_{12} \; = \; 0$$

A related cancellation in d=6

$$21\, I_{1/2} - I_{3/2} + 8\, I_A \; = \; 0$$

Type I d=10

$$\begin{aligned}
\mathrm{I}_{12} = & -\frac{1}{15}\mathrm{Tr}\,\mathrm{F}^6 + \frac{1}{24}\mathrm{Tr}\,\mathrm{F}^4\mathrm{Tr}\,\mathrm{R}^2 - \frac{\mathrm{Tr}\,\mathrm{F}^2}{960}(5(\mathrm{Tr}\,\mathrm{R}^2)^2 + 4\mathrm{Tr}\,\mathrm{R}^4) \\
& + \frac{\mathrm{N} - 496}{7560}\mathrm{Tr}\,\mathrm{R}^6 + \left(\frac{\mathrm{N} - 496}{5760} + \frac{1}{8}\right)\mathrm{Tr}\,\mathrm{R}^4\mathrm{Tr}\mathrm{R}^2 \\
& + \left(\frac{\mathrm{N} - 496}{13824} + \frac{1}{32}\right)(\mathrm{Tr}\,\mathrm{R}^2)^3
\end{aligned}$$

Cancelling ingenuity

Green and Schwarz proved a remarkable result

$$\mathrm{I}_{12} = (\mathrm{Tr}\,\mathrm{R}^2 + \mathrm{k}\,\mathrm{Tr}\,\mathrm{F}^2)\mathrm{X}_8$$

$$\mathrm{Tr}\,\mathrm{F}^6 = \frac{1}{48}\,\mathrm{Tr}\,\mathrm{F}^2\,\mathrm{Tr}\,\mathrm{F}^4 - \frac{1}{14400}\,(\mathrm{Tr}\,\mathrm{F}^2)^3$$

$$\mathrm{k} = \frac{-1}{30}$$

$$\mathrm{X}_8 = \frac{1}{24}\mathrm{Tr}\,\mathrm{F}^4 - \frac{1}{7200}(\mathrm{Tr}\,\mathrm{F}^2)^2 - \frac{1}{240}\mathrm{Tr}\,\mathrm{F}^2\,\mathrm{Tr}\,\mathrm{R}^2 + \frac{1}{8}\mathrm{Tr}\,\mathrm{R}^4 + \frac{1}{32}(\mathrm{Tr}\,\mathrm{R}^2)^2 \,.$$

This is the case for:

$Spin(32)/Z_2$

$E_8 \times E_8$

Heterotic Strings

Then by modifying the transformation rules and adding some specific counterterms, the anomaly cancels

$$\mathrm{S}_c = \int \left[4\left(\mathrm{Q}_{3L} - \frac{1}{30}\,\mathrm{Q}_{3y}\right)\mathrm{X}_7 - 6\mathrm{B}\,\mathrm{X}_8\right]$$

$$\mathrm{dX}_7 = \mathrm{X}_8 \,.$$

Local anomalies is not the end...

Once the local anomalies have been cancelled, we may start worrying about global anomalies. This is a rather subtle subject, where the index theorem, the spectral flow and a good deal of differential topology are needed in order to test the theory under these transformations. This is Witten's land for the most part.

$S^{2n-1} \times R$ $\qquad\qquad$ $\mathcal{G} \equiv \{g : S^{2n-1} \to G\}$

The group of gauge transformations corresponds to transformations that are continuously related to the identity. These are the gauge or diff that are dealt with by Gauss's law. This space however need not be simply connected. If:

$$\pi_1(\mathcal{G}) \neq \emptyset$$

Gauss's law can be integrated locally, but not globally, it may turn out that the partition function of the theory vanishes identically. This is the case the famous Witten anomaly in d=4 G=SU(2) $\pi_4(SU(2)) = Z_2$

The computations in the gravitational case are rather involved, and it is remarkable that all theories that are free of local gauge and gravity anomalies, do not suffer from global diff anomalies (Witten)

We have not exhausted the fauna of possible anomalies in string and supergravity theories, specially if we consider the contributions of branes, however the same technology applies to them, although the form of the relevant anomaly polynomials includes contributions also from Chern-Simons forms and RR-fields. K-theory is quite useful

Similar arguments apply to orbifold field theories. They imply Horava-Witten, and Green-Schwarz anomaly analysis. Useful in the treatment of theories with large extra dimensions

Conclusions

It provides an excellent arena to learn the subtle properties of gauge and gravitational theories with chiral matter

The cancellation of anomalies in supergravity theories puts very stringent contraints on the theories that are consistent

Apart from the original applications, they are needed still when trying to obtain low-dimensional realistic models in theories with flux compactifications, large extra dimensions, etc where the contribution of branes wrapped around nontrivial cycles is important.