Les Houches Lectures on Cosmology and Fundamental Theory

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1. Review of Standard Cosmology

As one discusses the universe at the largest observed scales one finds that it is spatially uniform. So one describes it using a spatially uniform metric

$$ds^2 = -dt^2 + a(t)^2 dx_i dx_i = a^2[-d\eta^2 + dx_i dx_i] \qquad (1.1)$$

where t is proper time and η is called conformal time. Notice that the second parametrization is useful for thinking about the Penrose diagram since we have written the metric as the flat space metric. We will find however, that in general we will only get a region of flat space since a can diverge at certain values of η. Note that the universe is uniform and isotropic in space but it is not uniform in time, it was different in the past. In writing the metric (1.1) I have assumed that the universe is spatially flat, which is in good agreement with current observations, but one could have imagined also spatial sections with constant positive or negative curvature which would also be homogeneous and isotropic. From now on I will mainly discuss the flat case. We can define the expansion rate

$$H = \frac{1}{R_H} \equiv \frac{\dot{a}}{a} \qquad (1.2)$$

where the dot is a derivative with respect to proper time. We have also introduced a quantity called the Hubble radius. We will later see that the Hubble radius is a length scale which characterizes the range of influence of the physics that is happening at a certain time. We will discuss this in more detail below.

The scale factor also characterizes the redshift that a photon emitted at time t and observed at a later time t_0 suffers via

$$(1 + Z) = \frac{\lambda_0}{\lambda_t} = \frac{a(0)}{a(t)} \qquad (1.3)$$

The evolution of the universe is determined by Einstein's equations after we make some statement about the matter distribution. It is possible to assume that the matter distribution is given by a perfect fluid with a stress tensor of the form $T^\mu_\nu \sim diag(\rho, -p, -p, -p)$ characterized by the density and pressure. We include a possible cosmological constant as a contribution to the stress tensor.

CONTINUE

2. Generation of fluctuations during inflation

The computation of primordial fluctuations that arise in inflationary models was first discussed in [1][2][3][4][5][6] and was nicely reviewed in [7].

The starting point is the Lagrangian of gravity and a scalar field which has the general form

$$S = \frac{1}{2}\int \sqrt{g}[R - (\nabla\phi)^2 - 2V(\phi)] \tag{2.1}$$

up to field redefinitions. We have set $M_{pl}^{-2} \equiv 8\pi G_N = 1$[1], the dependence on G_N is easily reintroduced.

The homogeneous solution has the form

$$ds^2 = -dt^2 + e^{2\rho(t)}dx_i dx_i = e^{2\rho}(-d\eta^2 + dx_i dx_i) \tag{2.2}$$

where η is conformal time. The scalar field is a function of time only. ρ and ϕ obey the equations

$$\begin{aligned} 3\dot\rho^2 =& \frac{1}{2}\dot\phi^2 + V(\phi) \\ \ddot\rho =& -\frac{1}{2}\dot\phi^2 \\ 0 =& \ddot\phi + 3\dot\rho\dot\phi + V'(\phi) \end{aligned} \tag{2.3}$$

The Hubble parameter is $H \equiv \dot\rho$. The third equation follows from the first two. We will make frequent use of these equations.

If the slow roll parameters are small we will have a period of accelerated expansion. The slow roll parameters are defined as

$$\begin{aligned} \epsilon &\equiv \frac{1}{2}\left(\frac{M_{pl}V'}{V}\right)^2 \sim \frac{1}{2}\frac{\dot\phi^2}{\dot\rho^2}\frac{1}{M_{pl}} \\ \eta &\equiv \frac{M_{pl}^2 V''}{V} \sim -\frac{\ddot\phi}{\dot\rho\dot\phi} + \frac{1}{2}\frac{\dot\phi^2}{\dot\rho^2}\frac{1}{M_{pl}} \end{aligned} \tag{2.4}$$

where the approximate relations hold when the slow roll parameters are small.

We now consider small fluctuations around the solution (2.3). We expect to have three physical propagating degrees of freedom, two from gravity and one from the scalar field. The scalar field mixes with other components of the metric which are also scalars under

[1] Note that this definition of M_{pl} is different from the definition that some other authors use (including Planck).

$SO(2)$ (the little group that leaves $\vec{k}$ fixed). There are four scalar modes of the metric which are δg_{00}, δg_{ii}, $\delta g_{0i} \sim \partial_i B$ and $\delta g_{ij} \sim \partial_i \partial_j H$ where B and H are arbitrary functions. Together with a small fluctuation, $\delta\phi$, in the scalar field these total five scalar modes. The action (2.1) has gauge invariances coming from reparametrization invariance. These can be linearized for small fluctuations. The scalar modes are acted upon by two gauge invariances, time reparametrizations and spatial reparametrizations of the form $x^i \to x^i + \epsilon^i(t,x)$ with $\epsilon^i = \partial_i \epsilon$. Other coordinate transformations act on the vector modes[2]. Gauge invariance removes two of the five functions. The constraints in the action remove two others so that we are left with one degree of freedom.

In order to proceed it is convenient to work in the ADM formalism. We write the metric as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.5}$$

and the action (2.1) becomes

$$S = \frac{1}{2}\int \sqrt{h}\left[NR^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - Nh^{ij}\partial_i\phi\partial_j\phi\right] \tag{2.6}$$

Where

$$\begin{aligned} E_{ij} &= \frac{1}{2}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i) \\ E &= E^i_i \end{aligned} \tag{2.7}$$

Note that the extrinsic curvature is $K_{ij} = N^{-1}E_{ij}$. In the computations we do below it is often convenient to separate the traceless and the trace part of E_{ij}.

In the ADM formulation spatial coordinate reparametrizations are an explicit symmetry while time reparametrizations are not so obviously a symmetry. The ADM formalism is designed so that one can think of h_{ij} and ϕ as the dynamical variables and N and N^i as Lagrange multipliers. We will choose a gauge for h_{ij} and ϕ that will fix time and spatial reparametrizations. A convenient gauge is

$$\delta\phi = 0\ , \qquad h_{ij} = e^{2\rho}[(1+2\zeta)\delta_{ij} + \gamma_{ij}]\ , \qquad \partial_i\gamma_{ij} = 0\ , \qquad \gamma_{ii} = 0 \tag{2.8}$$

where ζ and γ are first order quantities. ζ and γ are the physical degrees of freedom. ζ parameterizes the scalar fluctuations and γ the tensor fluctuations. The gauge (2.8) fixes

[2] There are no propagating vector modes for this Lagrangian (2.1). They are removed by gauge invariance and the constraints. Vector modes are present when more fields are included.

the gauge completely at nonzero momentum. In order to find the action for these degrees of freedom we just solve for N and N^i through their equations of motion and plug the result back in the action. This procedure gives the correct answer since N and N^i are Lagrange multipliers. The gauge (2.8) is very similar to Coulomb gauge in electrodynamics where we set $\partial_i A_i = 0$, solve for A_0 through its equation of motion and plug this back in the action[3].

The equation of motion for N^i and N are the the momentum and hamiltonian constraints

$$\begin{aligned} &\nabla_i[N^{-1}(E^i_j - \delta^i_j E)] = 0 \\ &R^{(3)} - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - \dot\phi^2 = 0 \end{aligned} \tag{2.9}$$

where we have used that $\delta\phi = 0$ from (2.8). We can solve these equations to first order by setting $N^i = \partial_i \psi + N^i_T$ where $\partial_i N^i_T = 0$ and $N = 1 + N_1$. We find

$$N_1 = \frac{\dot\zeta}{\dot\rho}\ , \qquad N^i_T = 0\ , \qquad \psi = -e^{-2\rho}\frac{\zeta}{\dot\rho} + \chi\ , \qquad \partial^2\chi = \frac{\dot\phi^2}{2\dot\rho^2}\dot\zeta \tag{2.10}$$

In order to find the quadratic action for ζ we can replace (2.10) in the action and expand the action to second order. For this purpose it is not necessary to compute N or N^i to second order. The reason is that the second order term in N will be multiplying the hamiltonian constraint, $\frac{\partial L}{\partial N}$ evaluated to zeroth order which vanishes since the zeroth order solution obeys the equations of motion. There is a similar argument for N^i. Direct replacement in the action gives, to second order,

$$\begin{aligned} S =& \frac{1}{2}\int e^{\rho}(1+\zeta)(1+\frac{\dot\zeta}{\dot\rho})[-4\partial^2\zeta - 2(\partial\zeta)^2 - 2V] + \\ & e^{3\rho}(1+3\zeta)(1-\frac{\dot\zeta}{\dot\rho})[-6(\dot\rho+\dot\zeta)^2 - \frac{2}{3}(\partial^2\psi)^2 + \dot\phi^2] \end{aligned} \tag{2.11}$$

where we have neglected a total derivative which is linear in ψ. After integrating by parts some of the terms and using the background equations of motion (2.3) we find the final expression to second order[4]

$$S = \frac{1}{2}\int dt d^3x \frac{\dot\phi^2}{\dot\rho^2}[e^{3\rho}\dot\zeta^2 - e^{\rho}(\partial\zeta)^2] \tag{2.12}$$

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[3] As in electrodynamics in Coulomb gauge we will often find expressions which are not local in the spatial directions. In the linearized theory it is possible to define local gauge invariant observables where these non-local terms disappear.

[4] In order to compare this to the expression in [7] set $v = -z\zeta$ in (10.73) of [7].

No slow roll approximation was made in deriving (2.11). Note that naively the action (2.11) contains terms of the order $\dot{\zeta}^2$, while the final expression contains only terms of the form $\epsilon\dot{\zeta}^2$, so that the action is suppressed by a slow roll parameter. The reason is that the ζ fluctuation would be a pure gauge mode in de-Sitter space and it gets a non-trivial action only to the extent that the slow roll parameter is non-zero. So the leading order terms in slow roll in (2.11) cancel leaving only the terms in (2.12). A simple argument for the dependence of (2.12) on the slow roll parameters is given below.

Since (2.12) is describing a free field we just have a collection of harmonic oscillators. More precisely we expand

$$\zeta(t,x) = \int \frac{d^3k}{(2\pi)^3}\zeta_k(t)e^{i\vec{k}\vec{x}} \tag{2.13}$$

Each $\zeta_k(t)$ is a harmonic oscillator with time dependent mass and spring constants. The quantization is straightforward [8]. We pick two independent classical solutions $\zeta^{cl}_k(t)$ and $\zeta^{cl*}_k(t)$ of the equations of motion of (2.12)

$$\frac{\delta L}{\delta\zeta} = -\frac{d\left(e^{3\rho}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}_k\right)}{dt} - \frac{\dot{\phi}^2}{\dot{\rho}^2}e^{\rho}k^2\zeta_k = 0 \tag{2.14}$$

Then we write

$$\zeta_{\vec{k}}(t) = \zeta^{cl}_k(t)a^\dagger_{\vec{k}} + \zeta^{cl*}_k(t)a_{-\vec{k}} \tag{2.15}$$

where a and $a^\dagger$ are some operators. Demanding that $a^\dagger$ and a obey the standard creation and annihilation commutation relations we get a normalization condition for ζ^{cl}_k. Different choices of solutions are different choices of vacua for the scalar field. The comoving wavelength of each mode $\lambda_c \sim 1/k$ stays constant but the physical wavelength changes in time. For early times the ratio of the physical wavelength to the Hubble scale is very small and the mode feels it is in almost flat space. We can then use the WKB approximation to solve (2.14) and choose the usual vacuum in Minkowski space. When the physical wavelength is much longer than the Hubble scale

$$\lambda_{phys}H = \frac{\dot{\rho}e^{\rho}}{k} \gg 1 \tag{2.16}$$

the solutions of (2.14) go rapidly to a constant.

A useful example to keep in mind is that of a massless scalar field f in de-Sitter space. In that case the action is $S = \frac{1}{2}\int H^{-2}\eta^{-2}[(\partial_\eta f)^2 - (\partial f)^2]$ and the normalized classical solution, analogous to ζ^{cl}_k, corresponding to the standard Bunch Davies vacuum is [8]

$$f^{cl}_k = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} \tag{2.17}$$

where we are using conformal time which runs from $(-\infty, 0)$. Very late times correspond to small $|\eta|$ and we clearly see from (2.17) that f^{cl} goes to a constant. Any solution, including (2.17), approaches a constant at late times as $\eta^2 \sim e^{-2\rho}$, which is exponentially fast is physical time. In de-Sitter space we can easily compute the two point function for this scalar field and obtain[5]

$$\begin{aligned}\langle f_{\vec{k}}(\eta) f_{\vec{k}'}(\eta)\rangle =&(2\pi)^3\delta^3(\vec{k}+\vec{k}')|f^{cl}_k(\eta)|^2 = (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{H^2}{2k^3}(1+k^2\eta^2) \\ \sim&(2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{H^2}{2k^3} \qquad \text{for} \quad k\eta \ll 1\end{aligned} \tag{2.18}$$

We now go back to the inflationary computation. If one knew the classical solution to the equation (2.14) the result for the correlation function of ζ can be simply computed as

$$\langle \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)\rangle = (2\pi)^3\delta^3(\vec{k}+\vec{k}')|\zeta^{cl}_k(t)|^2 \tag{2.19}$$

If the slow roll parameters are small when the comoving scale $\vec{k}$ crosses the horizon then it is possible to estimate the late time behavior of (2.19) by the corresponding result in de-Sitter space (2.18) with a Hubble constant that is the Hubble constant at the moment of horizon crossing. The reason is that at late times ζ is constant while at early times the field is in the vacuum and its wavefunction is accurately given by the WKB approximation. Since the action (2.12) also contains a factor of $\dot{\phi}/\dot{\rho}$ we also have to set its value to the value at horizon crossing, this factor only appears in normalizing the classical solution. In other words, near horizon crossing we set $\zeta = \frac{\dot{\phi}}{\dot{\rho}} f$ where f is a canonically normalized field in de-Sitter space. This produces the well known result

$$\langle \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)\rangle \sim (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{1}{2k^3}\frac{\dot{\rho}_*^2}{M_{pl}^2}\frac{\dot{\rho}_*^2}{\dot{\phi}_*^2} \tag{2.20}$$

where the star means that it is evaluated at the time of horizon crossing, i.e. at time t_* such that

$$\dot{\rho}(t_*)e^{\rho(t_*)} \sim k \ . \tag{2.21}$$

[5] In coordinate space the result for late times is $\langle f(x,t)f(x',t)\rangle sim - \frac{H^2}{(2\pi)^2}\log(|x-x'|/L)$ where is an IR cutoff which is unimportant when we compute differences in f as we do in actual experiments.

The dependence of (2.20) on t_* leads to additional momentum dependence. It is conventional to parameterize this dependence by saying that the total correlation function has the form k^{-3+n_s} where

$$n_s = k\frac{d}{dk}\log(\frac{\dot\rho_*^4}{\dot\phi_*^2}) \sim \frac{1}{\dot\rho_*}\frac{d}{dt_*}\log(\frac{\dot\rho_*^4}{\dot\phi_*^2}) = -2(\frac{\ddot\phi_*}{\dot\rho_*\dot\phi_*} + \frac{\dot\phi_*}{\dot\rho_*}) = 2(\eta - 3\epsilon) \qquad (2.22)$$

As it has been often discussed, after horizon crossing the mode becomes classical, in the sense that the commutator $[\dot\zeta, \zeta] \to 0$ exponentially fast. So for measurements which only involve ζ or $\dot\zeta$ we can treat the mode as a classical variable.

After the end of inflation the field ϕ ceases to determine the dynamics of the universe and we eventually go over to the usual hot big bang phase. It is possible to prove [5][7] that ζ remains constant outside the horizon as long as no entropy perturbations are generated and a certain condition on the off-diagonal components of the spatial stress tensor is obeyed[6]. These conditions are obeyed if the universe is described by a single fluid or by a single scalar field. We should mention that for a general fluid the variable ζ can be defined in terms of the three metric as above (2.8) in the comoving gauge where $T^0_i = 0$[7]. In the case of a scalar field this implies that $\delta\phi = 0$. This gauge is convenient conceptually since the variable ζ is directly a function appearing in the metric. We see that the variable ζ tells us how much the spatial directions have expanded in the comoving gauge, so that to linear order ζ determines the curvature of the spatial slices $R^{(3)} = 4k^2\zeta$ [10]. This variable ζ is very useful in order to continue through the end of inflation since it is defined throughout the evolution and it is constant outside the horizon. An intuitive way to understand why ζ is constant is to note that the conditions stated above imply that two observers separated by some distance see the universe undergoing precisely the same history. Outside the horizon (where we can set $k = 0$ in all equations) ζ is just a rescaling of coordinates and this rescaling is a symmetry of the equations.

[6] The condition is $\partial_i\partial_j(\delta T_{ij} - \frac{1}{3}\delta_{ij}T_{ll}) = 0$.

[7] For readers who are familiar with Bardeen's classic paper [9], we should mention that the gauge invariant definition of ζ is $\zeta = h + (\mathcal{H}^{-1}h' - A)\mathcal{H}^2/(\mathcal{H}^2 - \dot{\mathcal{H}})$ where $\mathcal{H} = \rho'$ and primes indicate derivatives with respect to conformal time and $h = H_L + H_T/3$ with $A,\ H_L,\ H_T$ defined in [9]. In circumstances where ζ is conserved then it also reduces to the definition in terms of Bardeen potentials in [5], [7] (actually $\zeta_{here} = -\zeta_{there}$). The gauge choice that makes $h = \zeta$ is $T^0_i = 0$ or, using the equations of motion, $\dot h = \dot\rho A$.

Other gauges can be more convenient in order to do computations in the slow roll approximation. A gauge that is particularly convenient is

$$\delta\phi \equiv \varphi(t,x) \ , \qquad h_{ij} = e^{2\rho}(\delta_{ij} + \gamma_{ij}) \ , \qquad \partial_i \gamma_{ij} = 0 \ , \qquad \gamma_{ii} = 0 \tag{2.23}$$

where we have denoted the small fluctuation of the scalar field by φ. In order to avoid confusion, from now on ϕ will denote the background value of the scalar field and φ will be its deviation from the background value. We expect that in this gauge the action will be approximately the action of a massless scalar field φ to leading order in slow roll. Indeed, we can check that the first order expressions for N and N^i are

$$N_{1\,\varphi} = \frac{\dot\phi}{2\dot\rho}\varphi \ , \qquad N^i_\varphi = \partial_i \chi \ , \qquad \partial^2\chi = \frac{\dot\phi^2}{2\dot\rho^2}\frac{d}{dt}\left(-\frac{\dot\rho}{\dot\phi}\varphi\right) \tag{2.24}$$

where the φ subindex reminds us that $N_{1\,\varphi}$, N^i_φ are computed in the gauge (2.23). We see that these expressions are subleading in slow roll compared to φ. So in order to compute the quadratic action to lowest order in slow roll it is enough to consider just the $(\nabla\varphi)^2$ term in the action (2.1) since V'' is also of higher order in slow roll. This is just the action of a massless scalar field in the zeroth order background. We can compute the fluctuations in φ in the slow roll approximation and we find a result similar to that of a scalar field in de-Sitter space (2.18) where the Hubble scale is evaluated at horizon crossing. After horizon crossing we can evaluate the gauge invariant quantity ζ. This is most easily done by changing the gauge to the gauge where $\varphi = 0$. This can be achieved by a time reparametrization of the form $\tilde t = t + T$ with

$$T = -\frac{\varphi}{\dot\phi} \tag{2.25}$$

where t is the time in the gauge (2.8) and $\tilde t$ is the time in (2.23). After the gauge transformation (2.25), we find that the metric in (2.23) becomes of the form in (2.8) with

$$\zeta = \dot\rho T = -\frac{\dot\rho}{\dot\phi}\varphi \tag{2.26}$$

Incidentally, this implies that χ in (2.24) is the same as χ in (2.10). So the correlation function for ζ can be computed as the correlation function for φ times the factor in (2.26). In order to get a result as accurate as possible we should perform the gauge transformation (2.26), just after crossing the horizon so that the factor in (2.26), is evaluated at horizon crossing leading finally to (2.20). In principle we could compute ζ from φ at any time. If we were to choose to do it a long time after horizon crossing we would need to take into account that φ changes outside the horizon. This would require evaluating the action (2.1) to higher order in the slow roll parameters. Of course, the dependence for φ outside the horizon is such that it precisely cancels the time dependence of the factor in (2.26) so that ζ is constant.

In summary, the computation is technically simplest if we start with the gauge (2.23) and we compute the two point function of φ after horizon exit and at that time compute the ζ variable which then remains constant. On the other hand the computation in the gauge (2.8) is conceptually simpler since the whole computation always involves the variable of interest which is ζ. In other words, the gauge (2.23) is more useful before and during horizon crossing while the gauge (2.8) is more useful after horizon crossing.

These last few paragraphs are basically simple argument presented in [3]. The computation of fluctuations of φ in de-Sitter produces fluctuations of the order $\varphi = \frac{H}{2\pi}$ and then this leads to a delay in the evolution by $\delta t = -\varphi/\dot{\rho}$ (see (2.25)) which in turn gives an additional expansion of the universe by a factor $\zeta = \dot{\rho}\delta t = -\frac{\dot{\rho}}{\dot{\phi}}\varphi$. This additional expansion is evaluated at horizon crossing in order to minimize the error in the approximation.

We now summarize the discussion of gravitational waves [11]. Inserting (2.8) in the action and focusing on terms quadratic in γ gives

$$S = \frac{1}{8}\int [e^{3\rho}\dot{\gamma}_{ij}\dot{\gamma}_{ij} - e^{\rho}\partial_l\gamma_{ij}\partial_l\gamma_{ij}] \tag{2.27}$$

As usual we can expand γ in plane waves with definite polarization tensors

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}(k)\gamma^s_{\vec{k}}(t)e^{i\vec{k}\vec{x}} \tag{2.28}$$

where $\epsilon_{ii} = k^i\epsilon_{ij} = 0$ and $\epsilon^s_{ij}(k)\epsilon^{s'}_{ij}(k) = 2\delta_{ss'}$. So we see that for each polarization mode we have essentially the equation of motion of a massless scalar field. As in our previous discussion, the solutions become constant after crossing the horizon. Computing the correlator just after horizon crossing we get

$$\langle\gamma^s_{\vec{k}}\gamma^{s'}_{\vec{k}'}\rangle = (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{1}{2k^3}\frac{2\dot{\rho}_*^2}{M_{pl}^2}\delta_{ss'} \tag{2.29}$$

where we reinstated the M_{pl} dependence. We can similarly define the tilt of the gravitational wave spectrum by saying that the correlation function scales as k^{-3+n_t} where n_t is given by

$$n_t = k\frac{d}{dk}\log\dot{\rho}_*^2 = -\frac{\dot{\phi}_*^2}{\dot{\rho}_*^2} = -2\epsilon \tag{2.30}$$

3. Models of inflation

In order to find a suitable inflation model we need to find a potential $V(\phi)$ which is flat enough so that $\epsilon, \eta < .05$ typically where

$$\begin{aligned} \epsilon &= \frac{1}{2}\left(\frac{M_{pl}V'}{V}\right)^2 \\ \eta &= \frac{M_{pl}^2 V''}{V} \\ M_p^{-2} &= 8\pi G_N \end{aligned} \tag{3.1}$$

We will also demand that the number of e-folds is around 50 or 60 where the precise number will depend on the details of reheating.

$$N_{eff} = \frac{1}{M_p^2}\int d\phi \frac{V}{V'} \tag{3.2}$$

In addition we should have appropriate reheating. I will not talk about this, but it is an important constraint. In addition we want the amplitude of the primordial fluctuations to be ....

TO BE CONTINUED

4. Remarks on AdS/CFT and dS/CFT

4.1. AdS/CFT

The computation that we did above was done with inflation in mind, but the same mathematical structure arises if one considers a single scalar field with a negative potential. In the slow roll case, the background will be a slightly deformed anti-de-Sitter space. This can be understood as a slightly deformed conformal field theory. In other words, a non-conformal field theory which is almost conformal. An incomplete list of references where situations of this sort were considered is [12][13][14][15][16][17]. Here we just mention a few results that are relevant for us, for a review see [18]. The variables γ^s that we used above are associated to the traceless components of the stress tensor while the variable ζ is associated to the trace of the stress tensor. More precisely, we have a coupling of the form $\int \frac{dk^3}{(2\pi)^3}[2\zeta_{-\vec{k}} T^i_i(\vec{k}) + 2\gamma^s_{-\vec{k}} T^s(\vec{k})]$, where T^s is defined by an expression similar to (2.28), with $\gamma \to T$. The fact that the definition of the scalar mode depends on the gauge is translated into the fact that in a field theory with a scale we can either change the dimensionfull coupling constant or we can change the overall scale in the metric. It is common to fix the coupling and change the metric, which then relates ζ to the trace of the stress tensor. Alternatively we can fix the metric and change the coupling constant. In the field theory we do not have two independent operators, we have only one operator related by the equation

$$2T^i_i = \beta_\lambda \mathcal{O} \tag{4.1}$$

where β_λ is the beta function for the coupling λ which appears in the field theory Lagrangian in front of the non-marginal operator as $\int \lambda \mathcal{O}$. The operator $\mathcal{O}$ is the one coupling to ϕ and the operator $2T^i_i$ couples to ζ. The factor of slow roll that relates the correlators of ζ and ϕ is precisely the factor β_λ appearing above [19].

From the computations in the previous sections we can also compute the correlation function of stress tensors and trace of the stress tensor in non-conformal theories. Depending on whether the slow roll approximation is valid or not we would need to use different formulae in those sections.

Two point functions of the trace of the stress tensor were considered in the AdS context in [15][14][16][17]. The derivation of the effective action for the corresponding field in AdS identical to the one in the dS context. Similarly, computations of three point functions in AdS can be done by performing minor modifications to the above formulae. We will be more explicit below.

Now we will review the AdS_4 computation (see [20] for a review) so that we can contrast it clearly to the dS_4 computation.

Let us consider a canonically normalized scalar field in Euclidean anti-Sitter space ($EAdS_4$) which is the same as hyperbolic space. The action is

$$S = R^2_{AdS} \int \frac{dz}{z^2} \frac{1}{2} [(\partial_z f)^2 + (\partial f)^2] \tag{4.2}$$

In order to do computations it will be necessary to consider classical solutions which go to zero for large z and obey prescribed boundary conditions at $z = z_c$. In momentum space these are

$$f_{\vec{k}} = f^0_{\vec{k}} \frac{(1 + kz)e^{-kz}}{(1 + kz_c)e^{-kz_c}} \ , \qquad k = |\vec{k}| \tag{4.3}$$

where $f^0_{\vec{k}}$ is the boundary condition we impose at $z = z_c$. One should then compute the action for this solution as a function of the boundary conditions. Inserting (4.3) into (4.2), integrating by parts and using the equations of motion we get

$$
\begin{aligned}
-S = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} R^2_{AdS} f^0_{-\vec{k}} \frac{1}{z_c^2} \frac{df_{\vec{k}}}{dz}\Big|_{z=z_c} &= -\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} R^2_{AdS} f^0_{-\vec{k}} f^0_{\vec{k}} \frac{k^2}{z_c(1+kz_c)} \\
\sim -\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} R^2_{AdS} f^0_{-\vec{k}} f^0_{\vec{k}} [\frac{k^2}{z_c} - k^3 + \cdots]
\end{aligned}
\tag{4.4}
$$

where the dots indicate terms of higher order in z_c. The term divergent in z_c is local in position space[8] and it is viewed as a divergence in the CFT which should be subtracted by a local counterterm. The term independent of z_c is non-local and gives rise to the two point function

$$
\langle \mathcal{O}(\vec{k}) \mathcal{O}(\vec{k}') \rangle_{EAdS} = \frac{\delta^2 Z}{\delta f^0_{\vec{k}} \delta f^0_{\vec{k}'}} \Big|_{f^0=0} \sim (2\pi)^3 \delta(\vec{k} + \vec{k}') R^2_{AdS} k^3 \tag{4.5}
$$

Where Z is the partition function of the Euclidean CFT which is approximated by $Z \sim e^{-S_{cl}}$, with S in (4.4).

4.2. dS-CFT

The dS/CFT was proposed [21][22] in analogy with AdS/CFT [23][24][25]. The dS/CFT postulates that the wavefunction of a universe which is asymptotically de-Sitter space can be computed in terms of a conformal field theory. More precisely, we have the formula

$$
\Psi[g] = Z[g] \tag{4.6}
$$

where the left hand side is the wavefunction of the universe for given three metric and the right hand side is the partition function of some dual conformal field theory. Actually the left hand side has rapidly oscillating pieces which can be expressed as local functions of the metric. We discard these pieces since they have the interpretation of local counterterms in the CFT. Here we are thinking of de-Sitter in flat slices (or Poincare coordinates) and we are imagining that all fields start in their life in the Bunch-Davies vacuum. This determines the wavefunction Ψ, at least in the context of perturbation theory. If we were considering global de-Sitter space then our discussion would be valid in a small patch in the future

[8] It is proportional to $\frac{1}{z_c} \int dx^3 \frac{1}{2} (\partial f^0)^2$.

where it can be approximated by the Poincare patch and the memory of the particular state that could have come from the far past is lost[9]. This point of view follows simply from the discussion in [22] in analogy with the standard discussion in Euclidean AdS where the same formula (4.6) is valid[10]. Nobody has found a concrete example of this duality and there are some suspicions that such a duality should not exist [30]. All we will do here is to do some computations on the gravity side in order to get some insight on the properties that this hypothetical CFT should have. If an example were found, then it would be a more powerful way of computing the wavefunction that semiclassical physics in de-Sitter or nearly de-Sitter space. Note that an observer living in eternal de-Sitter space will not be able to measure two point correlators such as (2.29) or the wavefunction (4.6) which involves distances much larger than the Hubble scale. Only so called "metaobservers" can measure these [22]. On the other hand if the universe is approximately de-Sitter for a while and then inflation ends and we go over to a radiation or matter dominated universe then these correlation functions become observable. In fact, we are metaobservers of the early inflationary epoch [31].

In [21][32] the relation between CFT operators and fields in the bulk was explored and various ways of defining operators were considered. It was found that given a scalar field in the bulk one could define two operators with two conformal dimensions differing by $\Delta_+ - \Delta_- = d$ where d is the dimension of the CFT. If the field we are considering in the bulk is the metric then it is clear that the corresponding operator is the stress tensor and it should have dimension d. Indeed we will see that this agrees precisely with what we expect from the prescription (4.6). Below we explain more precisely how this computation is related both to the inflationary computation (2.29) and the corresponding EAdS computation.

The first step is to compute the wavefunction as a function of a small fluctuation in a massless scalar field f. Since f is a free field, which is a collection of harmonic oscillators, all we need to do is to compute the wavefunction for these harmonic oscillators. We want

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9 The information of the state coming from the asymptotic past in global dS is contained on modes whose angular momenta, l, on the sphere is fixed, assuming the evolution is non-singular and in the context of perturbation theory. On the other hand, we are focusing on modes with $l \gg 1$ when we look at the Poincare patch.

10 In AdS/CFT formula (4.6) arises in the Euclidean context when we think of Euclidean time as the direction perpendicular to the boundary. Ψ can then be interpreted also as the Hartle-Hawking wave function [26]. See [27][28][29] for more on this point of view.

to compute the Schroedinger picture wavefunction at some time η_c as a function of the amplitude of the field f. The wavefunction is given by a sum over all paths ending with amplitude f and starting at the appropriate vacuum state. Since the action is quadratic this sum reduces to evaluating the action on the appropriate classical solution. We choose the standard Euclidean (Bunch-Davies) vacuum for the fields at early times. The classical solution obeying the appropriate boundary conditions is

$$f = f^0_{\vec{k}} \frac{(1 - ik\eta)e^{ik\eta}}{(1 - ik\eta_c)e^{ik\eta_c}} \tag{4.7}$$

The boundary conditions at large η are the ones that correspond to the statement that the oscillator is in its ground state, which can be defined adiabatically at early times. The condition is that the field should behaves as $e^{ik\eta}$ for $|\eta| \to \infty$. Note that $f_{-\vec{k}} \neq f^*_{\vec{k}}$ since the boundary condition we are imposing at early times is not a real condition on the field $f(\eta, x)$ [11]. This is one of the many ways to think about the harmonic oscillator wavefunction. When we evaluate the classical action on this solution we get

$$\begin{aligned} iS =& i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} R^2_{dS} \frac{1}{\eta_c^2} f^0_{-\vec{k}} \partial_\eta f_{\vec{k}}|_{\eta=\eta_c} = i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} R^2_{dS} \frac{k^2}{\eta_c(1 - ik\eta_c)} f^0_{-\vec{k}} f^0_{\vec{k}} \\ \sim & \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} R^2_{dS} [i \frac{k^2}{\eta_c} - k^3 + \cdots] f^0_{-\vec{k}} f^0_{\vec{k}} \end{aligned} \tag{4.8}$$

Note that we are dropping an oscillatory piece at $|\eta| \to \infty$ which is equivalent to slightly changing the contour of integration by $\eta \to \eta + i\epsilon$. This is the standard prescription for the vacuum state of a harmonic oscillator.

Notice that under

$$\eta = iz \ , \qquad R_{dS} = iR_{AdS} \tag{4.9}$$

the formulas (4.7) and (4.8) go into (4.3) and (4.4). The fact that (4.7) goes into (4.3) is intimately related to the statement that when the mode has short wavelength it is in the adiabatic vacuum. A consequence of this fact is that the two point function computed using dS_4 differs by a sign from the corresponding one in Euclidean AdS_4[12]. More explicitly we have

$$\langle \mathcal{O}(\vec{k}) \mathcal{O}(\vec{k}') \rangle_{dS_4} \equiv \left. \frac{\delta^2 Z}{\delta f^0_{\vec{k}} \delta f^0_{\vec{k}'}} \right|_{f^0=0} \sim (2\pi)^3 \delta(\vec{k} + \vec{k}') R^2_{dS} (-k^3) \tag{4.10}$$

---

[11] There is nothing wrong in considering a complex solution since all we are doing is to evaluate a functional integral by a saddle point approximation.

[12] In other dimensions there are extra is that appears in the relation.

We can easily check that this is the analytically continued version of (4.5) under (4.9).

Now let us understand the relation between the wavefunction computed in (4.8), which is $\Psi \sim e^{iS_{cl}}$ and the expectation values that appeared in our earlier discussion (2.18). Of course, the relation is that $\langle f^2 \rangle = \int \mathcal{D}f f^2 |\Psi(f)|^2$. We see that only the real piece in iS contributes. This has a finite limit at late times. The divergent pieces in (4.8) are all imaginary and do not contribute to the expectation value. The functional integration over f gives again (2.18). There is a crucial factor of 2 that comes from the square of the wavefunction, so that the relation between (2.18) and (4.10) is not a Legendre transform.

Our previous discussion focused on a scalar field and its corresponding operator $\mathcal{O}$. All that we have said above translates very simply for the traceless part of the metric and the traceless part of the stress tensor, since at the linearized level the action for the graviton in the traceless transverse gauge reduces to the action of a scalar field (2.27)(2.28). We are defining the stress tensor operator as

$$T_{ij}(x) \equiv \frac{\delta Z[h]}{\sqrt{h}\delta h^{ij}(x)} = \frac{\delta \Psi[h]}{\sqrt{h}\delta h^{ij}(x)} \tag{4.11}$$

which is a standard definition for a Euclidean field theory.[13] In this case the divergent term in (4.8) can be rewritten as $-i\frac{1}{2\eta_c}\int d^3x\sqrt{h}R^{(3)}$. Note that there is a factor of i. We want to remove this by a counterterm in the action of the Euclidean CFT. These factors of i are related to the fact that the renormalization group transformation in the CFT should be appropriately unitary since this RG transformation corresponds, in the context of perturbation theory, to unitary evolution of the wavefunction in the bulk. If we define the central charge of the CFT in terms of the two point function of the stress tensor we get a negative answer. This negative answer has a simple qualitative explanation. We know that the wavefunction in terms of small fluctuations is bounded, in the sense that it is of the form $e^{-\alpha|f|^2}$ with α positive, since each mode is a harmonic oscillator with positive frequency. This sign implies a negative sign for the two point function of the stress tensor. Similarly the trace of the stress tensor is related to the derivative of the wavefunction with respect to ζ.

[13] One might want to define it with an i so that $T_{jl} \equiv i\frac{\delta Z[h]}{\sqrt{h}\delta h^{jl}}$. This definition might be natural given that the counterterms (which represent the leading dependence of the wavefunction) are purely imaginary. In any case, it is trivial to go between both definitions.

After we understood the relation between two point functions of operators and expectation values of the corresponding fluctuations we can similarly understand the relation between three point functions. The wavefunction has the form

$$\begin{aligned}\Psi =&Exp\left[\frac{1}{2}\int d^3xd^3x'\langle\mathcal{O}(x)\mathcal{O}(x')\rangle f(x)f(x')+\right.\\ &\left.\frac{1}{6}\int d^3xd^3x'd^3x''\langle\mathcal{O}(x)\mathcal{O}(x')\mathcal{O}(x'')\rangle f(x)f(x')f(x'')\right]\end{aligned} \tag{4.12}$$

where we emphasized that derivatives of Ψ give correlation functions for the corresponding operators. The expectation values in momentum space are related by

$$\begin{aligned}\langle f_{\vec{k}}f_{-\vec{k}}\rangle' =&-\frac{1}{2Re\langle\mathcal{O}_{\vec{k}}\mathcal{O}_{-\vec{k}}\rangle'}\\ \langle f_{\vec{k}_1}f_{\vec{k}_2}f_{\vec{k}_3}\rangle' =&\frac{2Re\langle\mathcal{O}_{\vec{k}_1}\mathcal{O}_{\vec{k}_2}\mathcal{O}_{\vec{k}_3}\rangle'}{\prod_i(-2Re\langle\mathcal{O}_{\vec{k}_i}\mathcal{O}_{-\vec{k}_i}\rangle')}\end{aligned} \tag{4.13}$$

where the prime means that we dropped a factor of $(2\pi)^3\delta(\sum\vec{k})$. And Re indicates the real part. The factors of two come from the fact that we are squaring the wavefunction (4.12). Notice that this explains why $\langle TT\rangle\sim c$ while $\langle\gamma\gamma\rangle\sim 1/c$ where $c\sim -R_{dS}^2M_{pl}^2$.

Now consider three point functions. For example, consider the three point function of the traceless part of the stress tensor. This can be computed directly in dS_4 by inserting the classical solutions (4.7) into the cubic terms in the action. This gives

$$\langle T^{s_1}_{\vec{k}_1}T^{s_2}_{\vec{k}_2}T^{s_3}_{\vec{k}_3}\rangle = (2\pi)^3\delta^3(\sum\vec{k}_i)\frac{M_{pl}^2}{\dot{\rho}_*^2}(-\frac{1}{32})(\epsilon^{s_1}_{ii'}\epsilon^{s_2}_{jj'}\epsilon^{s_3}_{ll'}t_{ijl}t_{i'j'l'})I \tag{4.14}$$

where I is defined in . The result in $EAdS_4$ is the same as above except for a minus sign, which can be understood as coming from (4.9). When we perform this computation we need to drop a local divergent term which is proportional to $\frac{-i}{2\eta_c}\sqrt{h}R^{(3)}$. We did not have any divergence in due to the fact that we were computing the square of the wavefunction while in (4.14) we are computing the third derivative of the wavefunction. Of course, we can compute directly (4.14) from using (4.13). So in order to compute three point functions of the stress tensor in the hypothetical three dimensional field theory corresponding to a nearly dS_4 spacetime all we need to do is apply formula (4.13) to our results in section four. To go to the corresponding expectation values in $EAdS_4$ we just need to multiply all dS_4 results by a minus sign which comes from $R_{dS}^2\to -R_{AdS}^2$ and all correlators of the stress tensor have such a factor in front in the tree level gravity approximation.

Some of the points we explained above are specific to the four dimensional dS_4 case. The situation in dS_5 is rather interesting. The computation of fluctuations for a massless scalar field gives, outside the horizon,

$$\langle f_{\vec{k}} f_{\vec{k}'} \rangle \sim H^3 (2\pi)^4 \delta(\vec{k}+\vec{k}') \frac{4}{\pi} \frac{1}{k^4} \ , \qquad\qquad H = R_{dS}^{-1} \tag{4.15}$$

On the other hand the wavefunction $\Psi \sim e^{iS}$ has the form

$$iS = -\frac{i}{2} R_{ds}^3 \int \frac{d^4k}{(2\pi)^4} f^0_{\vec{k}} f^0_{-\vec{k}} [\frac{k^2}{2\eta_c^2} - \frac{1}{4} k^4 \log(-\eta_c k) - i\frac{\pi}{8} k^4 + \alpha k^4] \tag{4.16}$$

where α is a real number. Note that the only term contributing to (4.15) is the real term proportional to k^4. All other terms are purely imaginary. From (4.16) we can compute the non-local contribution to the two point function which gives

$$\langle \mathcal{O}(\vec{k}) \mathcal{O}(\vec{k}') \rangle_{dS_5} \sim (2\pi)^4 \delta(\vec{k}+\vec{k}') i R_{ds}^3 \frac{1}{4} k^4 \log k \tag{4.17}$$

The $EAdS_5$ answer is given by the analytic continuation (4.9). Notice that the i is due to the fact that we have an odd number of powers of R_{dS} and is consistent with the fact that the logarithmic term in the wavefunction is purely imaginary. For the stress tensor this gives an imaginary central charge and imaginary three point functions. It is rather interesting that the two point function (4.15) is related to a local term in the wavefunction, namely the term proportional to k^4, which is the only real term. In other words, the non-local piece in the wavefunction which determines the stress tensor seems unrelated to the local piece which determines the expectation value of the fluctuations. In other words, dS_5/CFT_4 would tell us how to compute the non-local piece in the wavefunction but will give us no information on the local piece. On the other hand from the inflationary point of view we would be interested in computing (4.15) which depends on the local part of the wavefunction, or the partition function of the CFT. Maybe in dS/CFT we are only allowed to use imaginary counterterms, then the field theory should be such that it allows the computation of the finite real local parts in the effective action. Note that the real term in (4.16) arises in the analytic continuation (4.9) from the term in the $EAdS_5$ wavefunction that is proportional to $k^4 \log(z_c k) \to -\frac{\pi}{2} k^4 + k^4 \log(-\eta_c k)$. So still, in some sense, the real part of the wavefunction (4.16) is intimately related to the non-local term in the wavefunction. It looks like this will be the situation in all odd dimensional dS spaces. The AdS_3 case studied in [21] seems special because there is no bulk propagating graviton. Stress tensor correlators in dS/CFT were also studied in [33][32].

Now let us reexamine the three point functions of stress tensor operators in the limit that one of the momenta is much smaller than the other two. We can then approximate the small momentum by zero. This zero momentum insertion of the stress tensor can be viewed as coming from an infinitesimal coordinate transformation. So we then know that the three point function is going to be given by the change of the two point function by this coordinate transformation. For example, an insertion of the trace of the stress tensor at zero momentum is equivalent to performing a rescaling of the coordinates without rescaling the mass scale of the theory. Then the three point function will be given by the scale dependence of the two point function. In other words

$$\langle 2T_i^i(0)\mathcal{O}(k)\mathcal{O}(k')\rangle = -k^i \frac{\partial}{\partial k^i}\langle \mathcal{O}(k)\mathcal{O}(k')\rangle \tag{4.18}$$

This is the reason why three point functions in this limit are proportional to the tilt of the scalar and tensor spectra respectively, see . There is a similar argument for the insertion of the traceless part of the stress tensor at zero momentum. Formula (4.18)is valid to all orders in slow roll.

Notice that in order to compute observable quantities from dS/CFT we will need to square the wavefunction and integrate over some range of values of the couplings and the metric of the space where the CFT is defined. In other words, in order to compute some physically interesting quantity it is not enough to consider the CFT on a fixed 3-manifold but over a range of three manifolds. This is the reason that expectation values in dS are not simply given by analytic continuation of the ones in EAdS [32] even though the wavefunction and correlation functions of operators are given by analytic continuation.[14] This makes it clear that even if dS/CFT is true there is no causality problem, one is not fixing the final state of the universe. One fixes it as an auxiliary step in order to compute the wavefunction but in order to compute probabilities we need to sum over all final boundary conditions. A slightly different integral over boundary conditions arises also in the $EAdS$ context when we consider certain relevant operators [34], or double trace operators [35]. In those cases this integration is the same as a change in the boundary condition. Note that this is *not* what happens in the dS context since we have the *square* of the wavefunction. One might conjecture that dS expectation values are given by two

[14] This analytic continuation is very clear for fields with $2mR_{dS_d} < d$. For fields with mass above this bound it is not so clear what the right prescription is. In this paper we focus our attention on the easy case.

CFTs (one for Ψ and one for Ψ^*) coupled together in some fashion. Note that then it is not clear if we should view the resulting object as a local field theory since in the resulting object is not defined on a fixed manifold since in order to compute expectation values we need to integrate over the three metric. The two copies of the CFT that we are talking about arise just at the future boundary, so these two copies are different than the two copies talked about in [22][21][32][33]. In global coordinates in addition we have the past boundary. Throughout this paper we have ignored the past boundary since we focused on distances larger than the Hubble scale but smaller than the total size of the spatial slice. In the Hartle and Hawking prescription for the wavefunction of the universe the past and future parts of the wavefunctions are complex conjugates of each other since the total wavefunction is real [26]. It is natural to suspect that these two pieces can be thought of as Ψ and Ψ^* in our discussion above.

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