Black Hole Microstate Counting and its Macroscopic Counterpart

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Black Holes

Black holes are classical solutions of the equations of motion of general theory of relativity.

Each black hole is surrounded by an event horizon that acts as a one way membrane.

Thus classically black holes behave as perfect black bodies at zero temperature.

In quantum theory a black hole behaves as a thermodynamic system with definite temperature, entropy etc.

$$T = \frac{\kappa}{2\pi}, \qquad S_{BH} = \frac{A}{4\,G_N}$$

Bekenstein, Hawking

κ: **acceleration due to gravity at the horizon of the black hole.**

A: Area of the event horizon

G_N: **Newton's gravitational constant**

Our units: $\hbar = c = k_B = 1$

For ordinary objects the entropy of a system has a microscopic interpretation.

We fix the macroscopic parameters (e.g. total electric charge, energy etc.) and count the number of quantum states – known as microstates – each of which has the same charge, energy etc.

d_{micro}: number of such microstates

Define microscopic (statistical) entropy:

$$S_{micro} = \ln d_{micro}$$

Question: Does the entropy of a black hole have a similar statistical interpretation?

We shall investige this question in string theory.

Which string theory?

Even though there is a unique string (M)-theory, it can exist in many different stable and metastable phases.

Without knowing precisely which phase of string theory describes the part of the universe we live in, we cannot directly compare string theory to experiments.

However there are some issues, like the issues involving black hole thermodynamics, which are universal, and hence can be addressed in any phase of string theory.

We shall choose to work with a convenient class of phases of string theory with lot of supersymmetry.

Many aspects of black hole thermodynamics have been studied in such supersymmetric phases of the theory, but we shall focus our attention on one particular aspect.

– entropy of the black hole in the zero temperature limit (supersymmetric, extremal black holes).

Advantage: Such a black hole is a stable state of the theory.

Strategy:

1. Identify a supersymmetric black hole carrying a certain set of electric charges $\{Q_i\}$ and magnetic charges $\{P_i\}$ and calculate its entropy $S_{BH}(Q, P)$ using the Bekenstein-Hawking formula.

Note: A general phase of string theory may have more that one Maxwell field and hence a black hole is characterized by multiple charges.

2. Identify the supersymmetric quantum states in string theory carrying the same set of charges.

These will include the fundamental strings but also other objects in string theory *e.g.* D-branes, Kaluza-Klein monopoles etc.

Calculate the number $d_{micro}(Q, P)$ of these states.

3. Compare $S_{micro} \equiv \ln d_{micro}(Q, P)$ with $S_{BH}(Q, P)$.

For a class of supersymmetric extremal black holes in string theory one indeed finds a match:

$$\mathbf{A/4G_N = \ln d_{micro}}$$

Strominger, Vafa, ...

This agreement also opens up new questions.

The Bekenstein-Hawking formula is an approximate formula that holds in classical general theory of relativity.

While string theory gives a theory of gravity that reduces to Einstein's theory when gravity is weak, there are corrections.

Thus the Bekenstein-Hawking formula for the entropy works well only when gravity at the horizon is weak.

Typically this requires the charges to be large.

The calculation on the microscopic side also simplifies when the charges are large.

Instead of doing exact counting of quantum states, we can use approximations which give the result for large charges.

For ordinary systems, thermodynamics provides an approximate description that becomes exact in the limit of large volume.

Is the situation with black holes similar, i.e. they only capture the information about the system in the limit of large charge and mass?

Or, does a black hole contain exact information about the ensemble of microstates that it describes?

For example:

1. Do black holes encode systematically corrections to the entropy due to finite size effect?

2. Are black holes capable of computing the distribution of global quantum numbers among the microstates?

In order to answer these question we need to:

1. Generalize the Bekenstein-Hawking formula to account for 'finite size corrections'

2. Find ways of calculating distribution of global quantum numbers among black hole microstates.

At the same time we must develop methods for microscopic counting that allows us to count the number of microstates precisely, and also find the distribution of global quantum numbers among the microstates.

In these lectures I shall try to review the progress on both fronts.

The role of index

The microscopic analysis is always done in a region of the moduli space where gravity is weak and hence the states do not form a black hole.

In order to be able to compare it with the results from the black hole side we must focus on quantities which do not change as we change the coupling from small to large value.

– needs appropriate supersymmetric index.

The appropriate index in D=4 is the helicity trace index.

Bachas, Kiritsis

Suppose we have a BPS state that breaks 4n supersymmetries.

$\rightarrow$ there will be 4n fermion zero modes (goldstino) on the world-line of the state.

Quantization of these zero modes will produce Bose-Fermi degenerate states.

Thus $\mathrm{Tr}(-1)^F$ vanishes.

Define: $B_{2n} = \frac{1}{(2n)!}\mathrm{Tr}(-1)^F(2h)^{2n} = \frac{1}{(2n)!}\mathrm{Tr}(-1)^{2h}(2h)^{2n}$

h: third component of angular momentum in rest frame.

For every pair of fermion zero modes $\mathrm{Tr}(-1)^F(2h)$ gives a non-vanishing result, leading to a non-zero B_{2n}.

Most of our studies will be on 1/4 BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories in D=4.

Preserves 4 out of 16 supersymmetries

$\Leftarrow$ breaks 12 supersymmetries.

Thus the relevant helicity trace index is B_6.

Note: Since on the microscopic side we compute an index, we must ensure that on the black hole side also we compute an index.

Otherwise we cannot compare the two results.

The simplest example: Heterotic string theory on T^6.

This theory has 28 gauge fields.

Thus a generic charged states is characterized by 28 dimensional electric charge vector Q and magnetic charge vector P.

The theory has T-duality symmetry $O(6, 22; \mathbb{Z})$ under which Q and P transform as vectors.

This allows us to define T-duality invariant bilinears in the charges:

$$Q^2, \qquad P^2, \qquad Q \cdot P$$

More general class of $\mathcal{N} = 4$ supersymmetric string theories can be constructed by taking orbifolds of heterotic string theory on T^6.

– CHL models **Chaudhuri, Hockney, Lykken**

These theories have $(\mathrm{r} + 6)$ U(1) gauge fields for different values of r.

Thus Q and P are (r+6) dimensional vectors.

In each of these theories, the index $B_6(Q, P)$ has been computed for a wide class of charge vectors (Q, P).

In each case the result is expressed as Fourier expansion coefficients of some well known functions $Z(\rho, \sigma, v)$:

$$B_6 = (-1)^{Q.P} \int d\rho \int d\sigma \int dv\, e^{-\pi i(\rho Q^2 + \sigma P^2 + 2v Q\cdot P)} Z(\rho, \sigma, v)$$

$Z(\rho, \sigma, v)$: **explicitly known in each of the examples, and transform as modular forms of certain weights under subgroups of $Sp(2, \mathbb{Z})$.**

Dijkgraaf, Verlinde, Verlinde; Shih, Strominger, Yin; David, Jatkar, A.S.; Dabholkar, Gaiotto, Nampuri; S. Banerjee, Srivastava, A.S.; Dabholkar, Gomes, Murthy; $\cdots$

General structure of $Z(\rho, \sigma, v)$:

$$Z(\rho,\sigma,v) = e^{-2\pi i(a\rho+b\sigma+cv)} \prod_{m,n,p} \left(1 - e^{2\pi i(m\rho+n\sigma+pv)}\right)^{-c(m,n,p)}$$

$c(m,n,p)$: known functions of m, n, p.

a,b,c: known constants

It is also possible to find the systematic expansion of B_6 for large charges.

In each case we find $B_6 < 0$.

$$\ln|B_6| = \pi\sqrt{Q^2P^2-(Q.P)^2} + f\left(\frac{Q.P}{P^2}, \frac{\sqrt{Q^2P^2-(Q.P)^2}}{P^2}\right) + \mathcal{O}(\text{charge}^{-2})$$

f: a known function.

Cardoso, de Wit, Kappeli, Mohaupt; David, Jatkar, A.S.

For example, for heterotic string theory compactified on a six dimensional torus,

$$f(\tau_1, \tau_2) = 12\ln\tau_2 + 24\ln\eta(\tau_1 + i\tau_2) + 24\ln\eta(-\tau_1 + i\tau_2)$$

η: Dedekind function

On special subspaces of the parameter space of the $\mathcal{N}=4$ supersymmetric string theories in (3+1) dimensions, the theory develops $\mathbb{Z}_{\mathrm{N}}$ discrete symmetry generated by an element g which commutes with supersymmetry.

Each theory has a certain set of allowed values of N.

Example: For heteroric on T^6 we can have N=1,2,3,4,5,6,7,8

On these special subspaces we can define the twisted index:

$$B_6^g = \frac{1}{6!}\mathrm{Tr}\left[(-1)^{2h}(2h)^6 g\right]$$

Like B_6, this index is also protected.

– contains information on the distribution of the $\mathbb{Z}_N$ charge among the members of the ensemble.

In each case we can calculate the twisted index B_6^g, and find that the result is again given by Fourier integrals of modular forms of subgroups of $Sp(2, \mathbb{Z})$.

$$B_6^g = (-1)^{Q.P} \int d\rho \int d\sigma \int dv\, e^{-\pi i(\rho Q^2 + \sigma P^2 + 2vQ\cdot P)} Z_g(\rho, \sigma, v)$$

Z_g are known functions.

Furthermore for large charges we find

$$B_6^g = \exp[\pi \sqrt{Q^2P^2 - (Q \cdot P)^2}/N + \cdots]$$

All these results provide us with the ‘experimental data’ to be explained by a ‘theory of black holes’.

Macroscopic analysis

Goal:

1. Develop tools for computing the entropy of extremal black holes beyond the large charge limit.

2. Apply it to black holes carrying the same charges for which we have computed the microscopic index.

3. Compare the macroscopic results with the microscopic results.

Step 0: Relate degeneracy to index.

A.S.; Dabholkar, Gomis, Murthy, A.S.

Bekenstein-Hawking formula gives us the degeneracy of microstates.

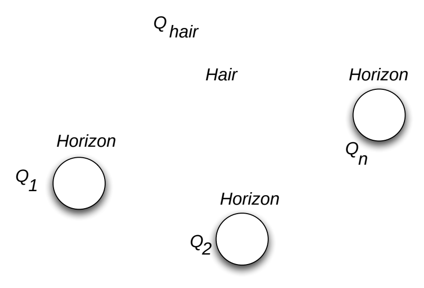
How can it be used to compute the index B_6?

In general the macroscopic degeneracy / index can have two kinds of contributions:

1. From the horizon.

2. From degrees of freedom living outside the horizon (hair). N. Banerjee, Mandal, A.S.

Example: The fermion zero modes associated with the broken supersymmetry generators are always part of the hair modes.



Q_i denotes both electric and magnetic charges of the i-th black hole.

We shall denote the degeneracy associated with the horizon degrees of freedom by d_{hor} and those associated with the hair degrees of freedom by d_{hair}.

d_{hair} can be calculated by explicitly identifying and quantizing the hair modes.

To leading order $d_{hor} \sim \exp[S_{BH}]$.

The total degeneracy:

$$\sum_n \sum_{\substack{\{\vec{Q}_i\}, \vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^n d_{hor}(\vec{Q}_i) \right\} d_{hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

Now let us compute B_6 for the same configuration.

$$B_6 = \frac{1}{6!}\,\mathrm{Tr}(-1)^{2h}(2h)^6 = \frac{1}{6!}\,\mathrm{Tr}(-1)^{h_{hor}+h_{hair}}(2h_{hor}+2h_{hair})^6$$

In four dimensions, supersymmetry $\to h_{hor} = 0$.

Thus

$$B_6 = \frac{1}{6!}\mathrm{Tr}(-1)^{h_{hair}}(2h_{hair})^6$$

$$\|$$

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i+\vec{Q}_{hair}=\vec{Q}}} \left\{\prod_{i=1}^n d_{hor}(\vec{Q}_i)\right\} B_{6;hair}(\vec{Q}_{hair};\{\vec{Q}_i\})$$

In $\mathcal{N} = 4$ supersymmetric string theories the contribution from multi-centered black holes is exponentially suppressed.

A.S.; Dabholkar, Guica, Murthy, Nampuri

Furthermore for single centered black holes often the only hair modes are the fermion zero modes.

In this case $Q_{hair} = 0$ and $B_{6;hair} = -1$.

Thus

$$B_6(Q) = -d_{hor}(Q)$$

Since $d_{hor}(Q) > 0$, we get $B_6 < 0$.

– agrees with the microscopic results.

Macroscopic formula for B_6:

$$\sum_n \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^n \vec{Q}_i+\vec{Q}_{hair}=\vec{Q}}} \left\{\prod_{i=1}^n d_{hor}(\vec{Q}_i)\right\} B_{6;hair}(\vec{Q}_{hair};\{\vec{Q}_i\})$$

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Computation of d_{hor}

To leading order $d_{hor}(Q) = \exp[S_{BH}]$.

Our goal will be to study corrections to this formula.

In string theory the Bekenstein-Hawking formula receives two types of corrections:

1. **Higher derivative (α') corrections in classical string theory.**
2. **Quantum (g_s) corrections.**

Of these the α' corrections are captured by Wald's modification of the Bekenstein-Hawking formula.

Furthermore for extremal black holes this formula takes a very simple form due to the AdS_2 factor in the near horizon geometry.

Reissner-Nordstrom solution in $D = 4$:

$$\begin{aligned} ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\ &\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\ &\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Define

$$2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\,\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}$$

and take $\lambda \to 0$ limit keeping r, t fixed.

$$ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

AdS_2 $\times$ S^2

Postulate: Any extremal black hole has an AdS_2 factor / $SO(2,1)$ isometry in the near horizon geometry.

– partially proved

Kunduri, Lucietti, Reall; Figueras, Kunduri, Lucietti, Rangamani

The full near horizon geometry takes the form $AdS_2 \times K$

K: some compact space that includes the S^2 factor.

Wald's general formula for computing higher deivative corrections to classical black hole entropy takes a very simple form for black holes with an AdS_2 factor in the near horizon geometry.

$$S_{wald} = 2\pi \left(q_i e_i - \sqrt{\det g_{AdS_2}}\, \mathcal{L}_{AdS_2} \right)$$

e_i: near horizon electric fields

q_i: electric charges conjugate to e_i

g_{AdS_2}: metric on AdS_2

$\sqrt{\det g_{AdS_2}}\mathcal{L}_{AdS_2}$: Classical Lagrangian density, evaluated on the near horizon geometry and integrated over K.

Quantum corrections:

What about quantum corrections?

Naive guess: apply Wald's formula again, but replacing the classical action by the 1PI action.

This will again give a simple algebraic method for computing the entropy.

This prescription is not complete since the 1PI action typically has non-local contribution due to massless states propagating in the loops.

Nevertheless this has been used to compute corrections to black hole entropy from local terms in the 1PI action with significant success.

Consider the CHL models obtained by $\mathbb{Z}_N$ orbifold of type IIB on $K3 \times S^1 \times \widetilde{S}^1$.

At tree level there are no corrections at the four derivative level, but at one loop these theories get corrections proportional to the Gauss-Bonnet term in the 1PI action.

$$\sqrt{-\det g}\Delta\mathcal{L}$$

$$= \psi(\tau, \bar{\tau}) \sqrt{-\det g} \left\{ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right\}$$

τ: modulus of the torus $(S^1 \times \widetilde{S}^1)$.

What is the effect of this term on the Wald entropy?

Adding this correction to the supergravity action one can calculate the correction to the entropy of a black hole in the CHL model.

Result for the Wald entropy

$$\pi\sqrt{\mathbf{Q}^2\mathbf{P}^2-(\mathbf{Q}.\mathbf{P})^2}$$
$$-\psi\left(\frac{\mathbf{Q}.\mathbf{P}}{\mathbf{P}^2},\sqrt{\frac{\mathbf{Q}^2\mathbf{P}^2-(\mathbf{Q}.\mathbf{P})^2}{\mathbf{P}^2}}\right)+\mathcal{O}\left(\frac{1}{\mathbf{Q}^2,\mathbf{P}^2,\mathbf{Q}.\mathbf{P}}\right)$$
$$\psi(\tau_1,\tau_2)\equiv(\mathbf{k}+\mathbf{2})\ln\tau_2+\ln\ \mathbf{g}(\tau_1+\mathbf{i}\tau_2)+\ln\ \mathbf{g}(-\tau_1+\mathbf{i}\tau_2)$$

$g(\tau)$: **a known function computed from the coefficient of the Gauss-Bonnet term.**

– agrees exactly with the result for $\ln|\mathbf{B}_6(\mathbf{Q},\mathbf{P})|$ calculated in the microscopic theory to order charge^0.

We shall now turn to the full quantum computation of d_{hor} from the macroscopic side.

In the classical limit d_{hor} should reduce to $\exp[S_{wald}]$.

The main tool: AdS_2/CFT_1 correspondence.

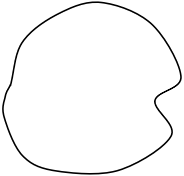
Steps for computing d_{hor}

1. Consider the euclidean AdS_2 metric:

$$\begin{aligned} ds^2 &= v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad 1 \le r < \infty, \quad \theta \equiv \theta + 2\pi \\ &= v(\sinh^2\eta\, d\theta^2 + d\eta^2), \quad r \equiv \cosh\eta, \quad 0 \le \eta < \infty \end{aligned}$$

Regularize the infinite volume of AdS_2 by putting a cut-off $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

This makes the AdS_2 boundary have a finite length L.



2. Define Z_{AdS_2}: Path integral over string fields in the euclidean near horizon background geometry weighted by

$$\exp[-\mathbf{Action} - i q_k \oint_{\partial(AdS_2)} d\theta\, A_\theta^{(k)}]$$

$\{q_k\}$: electric charges carried by the black hole under the U(1) gauge field $A^{(k)}$.

3. By AdS_2/CFT_1 correspondence:

$$Z_{AdS_2} = Z_{CFT_1}$$

$$Z_{CFT_1} = Tr(e^{-LH}) = d_0\, e^{-L\, E_0}$$

H: Hamiltonian of dual CFT_1 at the boundary of AdS_2.

(d_0, E_0): (degeneracy, energy) of the states of CFT_1.

$$Z_{AdS_2} = Z_{CFT_1} = d_0\, e^{-L\, E_0}$$

What is CFT_1?

– must be the quantum mechanics obtained by taking the infrared limit of the brane system describing the black hole.

This consists of a finite dimensional Hilbert space, consisting of the ground states of the brane system in a given charge sector.

Thus d_0 is the number of ground states of the black hole.

This suggests that we identify d_{hor} with d_0.

4. Thus we can define d_{hor} by expressing Z_{AdS_2} as

$$Z_{AdS_2} = e^{CL} \times d_{hor} \quad \text{as } L \to \infty$$

C: A constant

d_{hor}: 'finite part' of Z_{AdS_2}.

With this definition d_{hor} calculates d_0, i.e. the degeneracy of the dual CFT_1.

Note: Near the boundary of AdS_2, the θ indepndent solution to the Maxwell's equation has the form:

$$A_r = 0, \quad A_\theta = C_1 + C_2 r$$

C_1 (chemical potential) represents normalizable mode

C_2 (electric charge) represents non-normalizable mode

$\rightarrow$ the path integral must be carried our keeping C_2 (charge) fixed and integrating over C_1 (chemical potential).

Thus the AdS_2 path integral computes the entropy in the microcanonical ensemble.

This is also the reason why we need to insert the boundary term $\exp[-iq_k \oint_{\partial(AdS_2)} d\theta\, A_\theta^{(k)}]$ in the path integral.

Consistency check:

In the classical limit

$$\begin{aligned} Z_{AdS_2} &= \exp[-\text{Action} - iq_k \oint_{\partial(AdS_2)} d\theta\, A_\theta^{(k)}]\Big|_{\text{classical}} \\ &= \exp\left[2\pi\left(q_i e_i - \sqrt{\det g_{AdS_2}}\,\mathcal{L}_{AdS_2}\right) + CL\right] \\ &= \exp\left[S_{wald} + CL\right] \end{aligned}$$

Thus $d_{hor} = \exp[S_{wald}]$ in the classical limit.

Applications

1. Logarithmic corrections to the black hole entropy

– arises from one loop contribution to the path integral from massless fields.

– requires finding the eigenvalues of the kinetic operator in the near horizon geometry and then calculating the determinant.

Example: A free scalar with standard kinetic term gives a contribution to $\ln d_{hor}$ of the form:

$$-\frac{1}{180}\ln A$$

For the quarter BPS black holes in $\mathcal{N} = 4$ the calculation is straightforward but requires calculating eigenvalues of the kinetic operator after taking into account the mixing between various fields.

So far the contribution from the matter multiplet fields have been calculated.

Result: all contributions cancel **Gupta, S. Banerjee, A.S.**

– agrees with the microscopic results for which there is no logarithmic contribution that depends on the number of matter multiplets (r).

Second application: Twisted index

Suppose we want to compute the index

$$B_6^g = \frac{1}{6!}\,\mathrm{Tr}\left[(-1)^{2h}\,(2h)^6\,g\right]$$

g: some $\mathbb{Z}_N$ symmetry generator.

After separating out the contribution from the hair degrees of freedom, we see that the relevant quantity associated with the horizon is

$$-\mathrm{Tr}_{hor}((-1)^{2h_{hor}}g) = -\mathrm{Tr}_{hor}(g)$$

What macroscopic computation should we carry out?

By following the logic of AdS/CFT correspondence we find that we need to again compute the partition function on AdS_2, but this time with a g twisted boundary condition on the fields under $\theta \to \theta + 2\pi$.

Other than this the asymptotic boundary condition must be identical to that of the attractor geometry since the charges have not changed

The 'finite part' of this partition function gives us $\mathrm{Tr}_{\mathrm{hor}}(g)$.

Recall AdS_2 metric:

$$ds^2 = v\left[(r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right] = v\left[\sinh^2\eta d\theta^2 + d\eta^2\right]$$

The circle at infinity, parametrized by θ, is contractible at the origin $r = 1$.

Thus a g twist under $\theta \to \theta + 2\pi$ is not admissible.

$\to$ the $AdS_2 \times S^2$ geometry is not a valid saddle point of the path integral.

Question: Are there other saddle points which could contribute to the path integral?

Constraints:

1. It must have the same asymptotic geometry as the $AdS_2 \times S^2$ geometry.

2. It must have a g twist under $\theta \to \theta + 2\pi$.

3. It must preserve sufficient amount of supersymmetries so that integration over the fermion zero modes do not make the integral vanish.

Beasley, Gaiotto, Guica, Huang, Strominger, Yin; N. Banerjee, S. Banerjee, Gupta, Mandal, A.S.

There are indeed such saddle points in the path integral, constructed as follows.

1. Take the original near horizon geometry of the black hole.

2. Take a $\mathbb{Z}_N$ orbifold of this background with $\mathbb{Z}_N$ generated by simultaneous action of

a) $2\pi/N$ rotation in AdS_2

a) $2\pi/N$ rotation in S^2 (needed for preserving SUSY)

c) g.

N. Banerjee, Jatkar, A.S.; A.S.; Pioline, Murthy

To see that this has the same asymptotic geometry as the attractor geometry we make a rescaling:

$$\theta \to \theta/N, \qquad r \to N\,r$$

The metric takes the form:

$$v\left((r^2 - N^{-2})d\theta^2 + \frac{dr^2}{r^2 - N^{-2}}\right)$$

Orbifold action: $\theta \to \theta + 2\pi$, $\phi \to \phi + 2\pi/N$, g

The g transformation provides us with the correct boundary condition.

The ϕ shift can be regarded as a Wilson line, and hence is an allowed fluctuation in AdS_2.

The classical action associated with this saddle point, after removing the divergent part proportional to the length of the boundary, is S_{wald}/N.

Thus the contribution to the twisted partition function B_6^g from this saddle point is

$$Z_g^{finite} = \exp\left[S_{wald}/N\right]$$

This is exactly what we have found in the microscopic analysis of the twisted index.

Conclusion

Quantum gravity in the near horizon geometry contains detailed information about not only the total number of microstates. but also finer details *e.g.* the $\mathbb{Z}_N$ quantum numbers carried by the microstates.

Thus at least for extremal black holes there seems to be an exact duality between

Gravity description $\Leftrightarrow$ Microscopic description

The gravity description contains as much information as the microscopic description, but in quite different way.