Properties of low energy graviton scattering amplitudes

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Cargese Summer School on
"String Theory: Formal Developments and Applications"
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I) Non-perturbative aspects of maximally supersymmetric closed string theory scattering amplitudes.

Low energy expansion in flat space in $D < 11$ dimensions.
Role of duality symmetries.

II) Relationship to multiloop supergravity quantum field theory.

Ultraviolet divergences in D dimensions.

III) Perturbative supergravity using pure spinor quantum mechanics.

Explicit evidence for UV divergence of form d^4R^4 in $D=4$ maximal supergravity.

Non-perturbative properties of four-point scattering in Type II string theory :

1) MBG, Jorge Russo, Pierre Vanhove,
"String theory dualities and supergravity divergences",
arXiv:1001.2535;

2) MBG, Jorge Russo, Pierre Vanhove,
"String theory dualities and supergravity divergences",
arXiv:1002.3805

3) MBG, Stephen Miller, Jorge Russo, Pierre Vanhove,
"Eisenstein series for higher-rank groups and string theory amplitudes",
arXiv:1004.0163

Related work by: Pioline, Obers, Kiritsis, Basu, Sethi, ...

Five-loop maximal supergravity and evidence for a seven-loop UV divergence in N=8 supergravity

4) Jonas Bjornsson, MBG
"5 loops in 24/5 dimensions",
arXiv:1004.2692

I) # Dualities of type II theory

on d=(10-D)-torus $\mathcal{T}^d$

Duality invariance implies relations between perturbative and nonperturbative terms in the S-matrix.

Non-trivial dependence on the moduli (scalar fields),

- in contrast to classical supergravity

Moduli space : space of scalar fields

discrete identification- breaks symmetry to discrete subgroup

maximal compact subgroup

$$G(\mathbb{Z})\backslash \underbrace{G(\mathbb{R})/K}$$

Scalar fields in maximally supersymmetric supergravity

e.g. Type IIB theory in D=10 dimensions $\quad SL2(\mathbb{Z})\backslash SL2(\mathbb{R})/O(2)$

U-duality groups for type II in D dimensions

on $\mathcal{T}^d$ $D = 10 - d$

Dimension D	$G_d(\mathbb{Z}) = E_{d+1}$
$10A$	1
$10B$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{Z})$

decompactification

. . . . . .

Four (super)-Graviton Scattering in type II

$$A_D(s,t,u) = A_D^{analytic}(s,t,u) + A_D^{nonan}(s,t,u)$$

local term in eff. action

contains massless thresholds
- nonlocal terms in eff. action
(IR divergent for $D \leq 4$).

$$A_D^{analytic}(s,t,u) = \mathcal{R}^4\, T_D(s,t,u)$$

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_4)^2, \qquad u = -(k_1+k_3)^2$$

$\mathcal{R}$ Linearised supercurvature - describes 256 physical states in supermultiplet.

Momentum expansion:

$$T_D(s,t,u) = \sum_{p,q} \mathcal{E}^{(D)}_{(p,q)}\, \sigma_2^p\, \sigma_3^q$$

Power series in

$$\sigma_2 = s^2 + t^2 + u^2$$

$$\sigma_3 = s^3 + t^3 + u^3$$

Coefficients are duality invariants functions of moduli

Duality - invariant effective IIB action

(Einstein frame) Einstein-Hilbert Higher-derivative interactions

$$S = \frac{1}{\ell_D^{D-2}} \int d^D x \sqrt{-G^{(D)}}\, R + S^{local} + \dots$$

$\sigma_2\, \mathcal{R}^4$ $\sigma_3\, \mathcal{R}^4$

$$S_D^{local} = \ell_D^{8-D} \int d^D x \sqrt{-G^{(D)}} \left(\mathcal{E}^{(D)}_{(0,0)}\, \mathcal{R}^4 + \ell_D^4\, \mathcal{E}^{(D)}_{(1,0)}\, \partial^4 \mathcal{R}^4 + \ell_D^6\, \mathcal{E}^{(D)}_{(0,1)}\, \partial^6 \mathcal{R}^4 + \dots \right)$$

1/2 BPS 1/4 BPS 1/8 BPS

Planck length

Is this the complete list of "protected" terms??

$$\ell_D^{D-2} = \ell_{D+1}^{D-1} \frac{1}{r_d}$$

$$\ell_s^4 = \ell_{10}^4\, g_s^{-1} = \ell_{10}\, e^{-\phi} = \ell_{10}\, \tau_2$$

(complex coupling $\tau = \tau_1 + i\tau_2$)

string length scale

$\mathcal{E}^{(D)}_{(p,q)}$ are E_{d+1} - invariant coefficients

functions of moduli

How may these be determined ??

Simple cases can be determined by supersymmetry:

Sketch:

Low energy expansion of amplitude:

$$\ell_D^{D-2}\, S = S^{(0)} + l_D^{6}\, S^{(6)} + l_D^{10}\, S^{(10)} + l_D^{12}\, S^{(12)} + \ldots$$

Classical action

Expand SUSY transformation on fields Φ:

$$\delta\, \Phi = (\delta^{(0)} + \ell_D^{6} \delta^{(6)} + \ell_D^{10} \delta^{(10)} + \ell_D^{12} \delta^{(12)} + \ldots)\, \Phi$$

SUSY requires :

$$\delta\, S = (\delta^{(0)} + \ell_D^{6} \delta^{(6)} + \ldots)(S^{(0)} + l_D^{6}\, S^{(6)} + \ldots) = 0$$

Closure of superalgebra :

$$\begin{aligned} [\delta\,,\delta]\,\Phi &= [\delta^{(0)} + \ell_D^{6} \delta^{(6)} + \ldots, \delta^{(0)} + \ell_D^{6} \delta^{(6)} + \ldots]\,\Phi \\ &= a \cdot \partial\, \Phi + \text{eqs. of motion} \end{aligned}$$

e.g. D=10 IIB

duality group $SL(2,\mathbb{Z})$ modulus $\tau = \tau_1 + i\tau_2$ $\tau_2 \equiv e^{-\phi} = g_B^{-1}$

coupling

i) For $\mathcal{E}^{(10)}_{(0,0)}\mathcal{R}^4$ $\mathcal{E}^{(10)}_{(1,0)}\,\sigma_2\,\mathcal{R}^4$

$S^{(6)}$ $S^{(10)}$

SUSY implies Laplace eigenvalue equation

$\Delta^{(10)} = \tau_2^2\partial_\tau\partial_{\bar\tau}$ $\longrightarrow$ $\boxed{\Delta\quad \mathcal{E}^{(10)}_{(p,0)} = s(s-1)\,\mathcal{E}^{(10)}_{(p,0)}}$ $s = p + \frac{3}{2}$

Invariant laplacian for $SL(2,\mathbb{R})/SO(2)$

$s = \frac{3}{2}$ for R^4 $s = \frac{5}{2}$ for $\sigma_2\,R^4$

Unique solution is nonholomorphic Eisenstein series

subject to moderate growth b.c.
(perturbation theory for large τ_2)

$$\mathcal{E}^{(10)}_{(p,0)} = E_s = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

$$E_s = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

two power behaved terms

$$\sim 2\zeta(2s)\tau_2^s + (\dots)\zeta(2s-1)\tau_2^{1-s} + \sum_{k\neq 0}\mu(k,s)\,\left(e^{2\pi i k\tau} + c.c.\right)\left(1 + O(\tau_2^{-1})\right)$$

TREE-level term

$(\tau_2 = g_s^{-1})$

GENUS-$(\mathbf{s} - \frac{1}{2})$ term

D-INSTANTON terms

Infinite no. of pert. Corrections-

non-zero Fourier modes.

Interesting measure

$\int d\tau_1$ projects onto "Constant Term" – zero Fourier mode – **power behaved terms**, kills instantons

- NO HIGHER LOOP perturbative terms
- Non-renormalization at higher loops

examples: $E_{\frac{3}{2}}\,\mathcal{R}^4$, $E_{\frac{5}{2}}\,s^2\,\mathcal{R}^4$

tree-level + one-loop tree-level + two-loop

ii) Higher order: $\mathcal{E}^{(10}_{(0,1)}\,\sigma_3\,\mathcal{R}^4 \sim \mathcal{E}^{(10)}_{(0,1)}\,\partial^6\,\mathcal{R}^4$

Inhomogeneous Laplace eigenvalue equation

R^4 source

$$\boxed{(\Delta_\tau - 12)\,\mathcal{E}^{(10)}_{(0,1)} = -E_{\frac{3}{2}}\,E_{\frac{3}{2}} = -\left(\mathcal{E}^{(10)}_{(0,0)}\right)^2}$$

Motivated by four-graviton scattering amplitude, and consistent with supersymmetry expectations (but not yet Derived from SUSY).

$$\delta\,S = \delta^{(0)}\,S^{(12)} + \delta^{(12)}\,S^{(0)} + \delta^{(6)}\,S^{(6)} + \ldots$$

Mixing of intermediate terms
responsible for presence of source term

Constant term (zero Fourier mode) contains:

Genus 0, 1, 2, 3 + instanton anti-instanton pairs

Note:

Non-renormalisation. Interactions of the form $\partial^{2k}\,\mathcal{R}^4$ have contributions for genus-h with $h \leq k$

(for $h > 1$)

Does this continue to hold for $k > 3$??

[comments later]

Eisenstein series for higher-rank duality groups

General Eisenstein series depends on $r = \mathrm{rank}\ G$ parameters $s_i \quad i = 1, \ldots, r$ (Selberg, Langlands)

But Eisenstein series for a maximal parabolic subgroup, $P(\beta)$, defined with respect to a simple root β, depends on only one non-zero parameter

$$s = s_\beta \neq 0 \qquad s_i = 0, \ (i \neq \beta)$$

$$E^G_\beta(g;\mu) = \sum_{\gamma = P(\mathbb{Z})\backslash G(\mathbb{Z})} e^{<\mu\,, H(\gamma g)>}$$

$P(\mathbb{Z}) = P(\mathbb{R}) \cap G(\mathbb{Z})$

parabolic subgroup

$H(g)$ Cartan subalgebra

Automorphise – sum over coset

$\mu = (\mu_1, \ldots, \mu_r)$ Vector parameterising Cartan torus e.g. GL(d) Parabolic subgroup of matrices of form :

$$P(n_1, \ldots, n_q) = \begin{pmatrix} U & * \\ 0 & a \end{pmatrix}$$

$d \times d$ matrix $\qquad U \in GL(d-1)$ $\qquad \beta$

is associated with Dynkin label $[0 \cdots 01]$

NOTATION:

$\mathbf{E}^G_{[0\cdots 0\,1\,0\cdots 0];s}$ = Maximal Parabolic Eisenstein series for Parabolic subgroup $P(\beta)$ associated with Dynkin label $[0 \cdots 0 \;1\; 0 \cdots 0]$

index

root β

[Standard $SL(2,\mathbb{Z})$ Eisenstein series: $E_s = \mathbf{E}^{SL(2)}_{[1];s}$]

e.g. SL(2,Z)

$$E_s^{SL(2)} = \sum_{\gamma \in N(Z)\backslash SL(2,Z)} \mathrm{Im}(\gamma(\tau))^s$$

$$N = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$\gamma(\tau) = \tfrac{a\tau+b}{c\tau+d}\,, \qquad a,b,c,d \in Z\,, \qquad ad-bc=1$$

e.g. SL(d,Z) :

$$\mathbf{E}^{SL(d)}_{[10\ldots0];s} = \sum_{\mathbf{m}\in\mathbb{Z}^d} \frac{1}{(m^i\, G_{ij}\, m^j)^s}$$

Epstein Series

dependence onSL(d)/SO(d) moduli

$$\mathbf{E}^{SL(d)}_{[01\ldots0];s} = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{Z}^d} \frac{1}{(m^{[i}n^{j]}\, G_{ik}G_{j\ell}\, m^{[k}n^{l]})^s}$$

antisymmetrised sums

. . . . . .

Lattice sums are much more subtle for other duality groups SO(5,5), E_6, E_7, E_8

Stephen Miller

Evaluate sums by "brute force" (Mathematica)

These series satisfy Laplace equation - quadratic Casimir

$$(\Delta_{G/K} - \lambda)\,\mathbf{E}^{G}_{[0\cdots 0\,1\,0\cdots 0];s} = 0$$

eigenvalue depends on $G,\ \beta,\ s$

Consistency conditions determine coefficients:

Laplace equations + boundary conditions

Laplace equations in various dimensions:

Decompactification of laplacian starting from D and using D=10 expressions as boundary conditions.

$$\Delta^{(D)} \equiv \Delta_{E_{11-D}/K_D}$$

R^4

$$\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} = 6\pi\,\delta_{D-8,0}$$

Singularity in D=8

"critical" dimensions

$\partial^4 R^4$

$$\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} = 40\zeta(2)\,\delta_{D-7,0}$$

Singularity in D=7

$\partial^6 R^4$

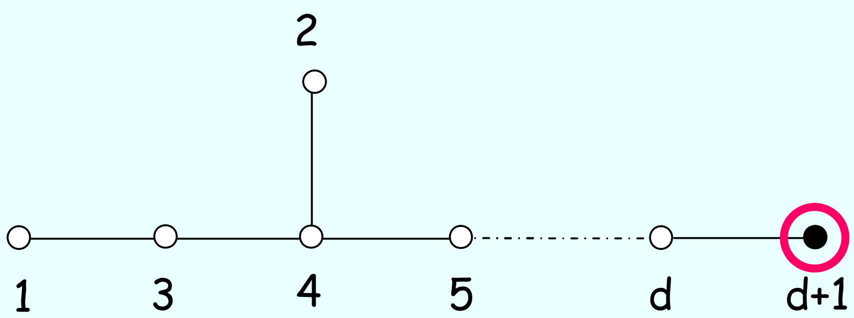
$$\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)} = -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 120\zeta(3)\,\delta_{D-6,0}$$

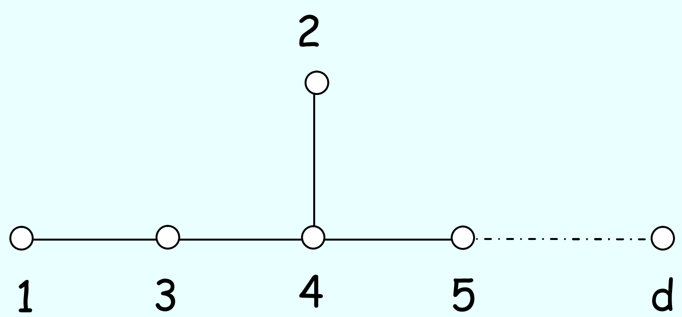
Singularity in D=6

Consistent degeneration limits of torus:

(i) Decompactify from D to D+1 dimensions

$E_{d+1} \rightarrow E_d \qquad (d = 10 - D) \qquad r_d \rightarrow \infty$





Power-behaved terms in $\int_{P(\alpha_d+1)} \mathcal{E}^{(D)}_{(p,q)}$ (in r_d)

zero Fourier mode
- kills instantons

"Constant term in parabolic subgroup" $P(\alpha_{d+1})$

(i) Decompactification of circle radius r_d

Consistency conditions :

$D = 10 - d$

terms proportional to r_d survive in D+1

terms needed to reproduce thresholds (unitarity)

$$R^4 \qquad \left(\frac{\ell_{D+1}}{l_D}\right)^{8-D} \int_{P(\alpha_d)} \mathcal{E}^{(D)}_{(0,0)} = \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{l_{D+1}}\right)^{8-D}$$

$$\partial^4 R^4 \quad \left(\frac{\ell_{D+1}}{l_D}\right)^{12-D} \int_{P(\alpha_d)} \mathcal{E}^{(D)}_{(1,0)} = \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(1,0)} + \left(\frac{r_d}{l_{D+1}}\right)^{6-D} \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{l_{D+1}}\right)^{12-D}$$

$$\partial^6 R^4 \quad \left(\frac{\ell_{D+1}}{l_D}\right)^{14-D} \int_{P(\alpha_d)} \mathcal{E}^{(D)}_{(0,1)} = \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(0,1)} + \left(\frac{r_d}{l_{D+1}}\right)^{4-D} \mathcal{E}^{(D+1)}_{(1,0)} + \left(\frac{r_d}{l_{D+1}}\right)^{8-D} \mathcal{E}^{(D+1)}_{(0,0)} + \left(\frac{r_d}{l_{D+1}}\right)^{14-D}$$

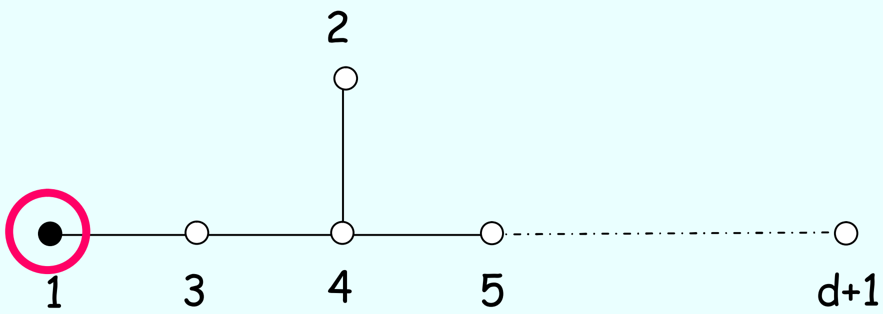
- Limited set of powers of r
- Note that all the D+1 coefficients are contained in $\partial^6 R^4$

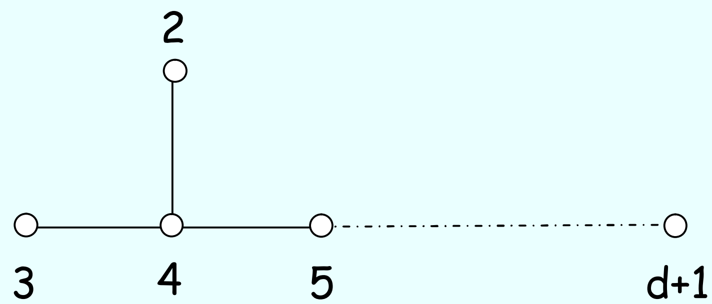
(ii) String perturbation expansion

weak
Coupling

$$\frac{1}{g_s^2} V_d = \frac{1}{y_D} \to \infty \qquad \text{fixed } \frac{r_d}{l_s}$$

$$E_{d+1} \to SO(d,d) \qquad \text{T-duality}$$





Perturbative terms in $\int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(p,q)}$

Powers of y_D correspond to world-sheet genus

Constant term in parabolic subgroup $P(\alpha_1)$

(ii) String perturbation expansion

Powers of y_D correspond to world-sheet genus

genus 0 1 2 3

R^4

$$\int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(0,0)} = \frac{l_s^{8-D}}{l_D^{8-D}} \left(\frac{2\zeta(3)}{y_D} + I^{(1)}_{(0,0)} \right)$$

2

$\partial^4 R^4$

$$\int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(1,0)} = \frac{l_s^{12-D}}{l_D^{12-D}} \left(\frac{2\zeta(4)}{y_D} + I^{(1)}_{(1,0)} + y_D I^{(2)}_{(1,0)} \right)$$

3

$\partial^6 R^4$

$$\int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(0,1)} = \frac{l_s^{14-D}}{l_D^{14-D}} \left(\frac{2\zeta^2(3)}{3y_D} + I^{(1)}_{(0,1)} + y_D I^{(2)}_{(0,1)} + y_D^2 I^{(3)}_{(0,1)} + O(e^{-4/y_D}) \right)$$

(iii) 11-dimensional supergravity limit $\mathcal{V}_{d+1} \to \infty$

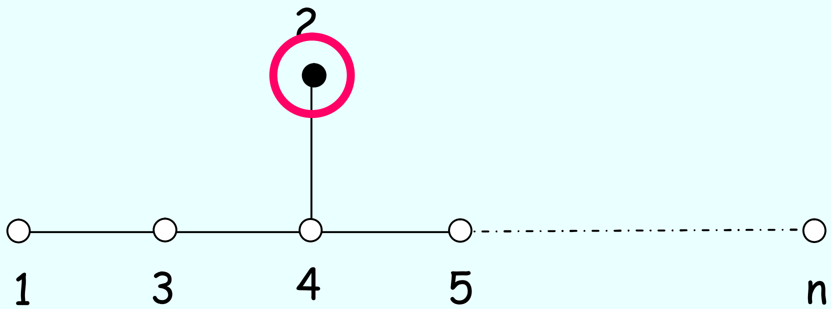
vol. of M-theory torus

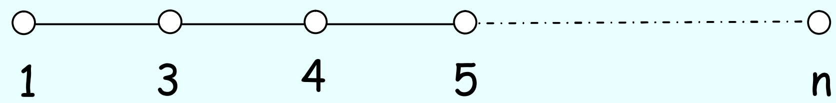
Large M-theory torus matches Feynman diagrams of eleven-dimensional supergravity

one, two loops

$$E_{d+1} \to SL(d+1)$$

geometric symmetry





Constant term in parabolic subgroup $P(\alpha_2)$

Perturbative terms in $\int_{P(\alpha_2)} \mathcal{E}^{(D)}_{(p,q)}$

SOLUTIONS to Laplace equations and b.c.'s:

e.g. $D=3,\ (E_8)$

MBG, Miller, Russo, Vanhove 2010
(Pioline 2010)

$$R^4 \qquad \mathcal{E}^{(3)}_{(0,0)} = \mathbf{E}^{E_8}_{[10000000];\frac{3}{2}}$$

$$\partial^4 R^4 \qquad \mathcal{E}^{(3)}_{(0,0)} = \mathbf{E}^{E_8}_{[10000000];\frac{5}{2}}$$

More complicated for

$$\mathcal{E}^{(3)}_{(0,1)}\,\partial^6\mathcal{R}^4$$

Perturbative expansions match in all three limits.

How can these expressions be interpreted in terms of sums over
1/2-BPS, 1/4–BPS, 1/8-BPS states ??

Solutions in all dimensions for $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$

$G_d = E_{d+1(d+1)}$	$\mathcal{R}^4$	$\partial^4\mathcal{R}^4$
$E_{8(8)}(\mathbb{Z})$	$\mathbf{E}^{E_8}_{[10000000];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_8}_{[1000000];\frac{5}{2}}$

(i) Decompactification of circle radius r

from D=3 to D=4

R^4 $$\mathbf{E}^{E_8}_{[10000000];\frac{3}{2}} = r^6\,\mathbf{E}^{E_7}_{[1000000];\frac{3}{2}} + r^{10}\,\frac{3\zeta(5)}{\pi}$$

+ instantons

R^4 coefficient in D=4

contribute to thresholds

$\partial^4 R^4$

$$\mathbf{E}^{E_8}_{[10000000];\frac{5}{2}} = r^{10}\,\frac{1}{2}\mathbf{E}^{E_7}_{[100000];\frac{5}{2}} + r^{12}\,\frac{\zeta(3)}{\pi}\,\mathbf{E}^{E_7}_{[100000];\frac{3}{2}} + r^{18}\,\frac{\pi^2\zeta(9)}{15}$$

$\partial^4 R^4$ coefficient in D=4

Note: Many terms have to vanish for this to work!!

$$\mathbf{E}^{E_8}_{[10000000];s} =$$
$$756r^{36} + 126r^{92-8s} + 576r^{60-4s} + 126r^{8s} + 576r^{14+4s}$$

(suppressing all coefficient E_7 Eisenstein series for simplicity)

(ii) D=3 String Perturbation Theory

e.g., D=3 (d=7) String coupling, $r^{-4} = y_3 = \frac{g_s^2}{V_7}$

R^4 $$\mathbf{E}^{E_8}_{[10000000];\frac{3}{2}} = r^{24}\, 2\zeta(3) + r^{20}\,\frac{3}{2\pi}\,\mathbf{E}^{SO(7,7)}_{[1000000];\frac{5}{2}}$$ + instantons

tree-level genus-one genus-two

$\partial^4 R^4$ $$\mathbf{E}^{E_8}_{[1000000];\frac{5}{2}} = r^{40}\,\zeta(5) + \frac{7}{24\pi}\, r^{36}\,\mathbf{E}^{SO(7,7)}_{[1000000];\frac{9}{2}} + \frac{2}{3} r^{32}\,\mathbf{E}^{SO(7,7)}_{[0000010];2}$$

Verify by explicit *genus-one* and *genus-two* calculations

Note nonrenormalisation conditions

(iii) 11-dimensional supergravity limit $\mathcal{V}_{d+1} \to \infty$

e.g., D=3 (d=7) $r^{1+d} = \mathcal{V}_{d+1}$ vol. of M-theory torus

R^4

$$\mathbf{E}^{E_8}_{[10000000];\frac{3}{2}} = r^{32}\, 4\zeta(2) + r^{30}\mathbf{E}^{SL(8)}_{[1000000];\frac{3}{2}}$$

+ instantons

$\partial^4 R^4$

$$\mathbf{E}^{E_8}_{[10000000];\frac{5}{2}} = \frac{1}{2} r^{50}\mathbf{E}^{SL(8)}_{[1000000];\frac{5}{2}} + \frac{2\zeta(3)}{\pi^2} r^{48}\mathbf{E}^{SL(8)}_{[0100000];2} + \frac{4\zeta(4)}{3}\, r^{54}\mathbf{E}^{SL(8)}_{[10000000];-\frac{1}{2}}$$

Precisely reproduced by one-loop and two-loop supergravity in eleven dimensions on (d+1)-torus.

very

NOTE: Many terms have to vanish for this to work!!

For generic s

$$\mathbf{E}^{E_8}_{[10000000];s} \sim 64r^{6s+1} + 448r^{41-2s} + 280r^{4(s+2)} + 448r^{2(s+9)} + r^{92-8s} + r^{8s} + 64r^{70-6s} + 280r^{54-4s} + 14r^{34} + 560r^{28}$$

(setting all coefficient SL(8) Eisenstein series = 1 for simplicity)

All but 2 or 3 vanish for s=3/2 or 5/2

II) Determining Supergravity UV divergences

MBG, Russo, Vanhove 2010

"Critical" Cases

Maximal SUGRA has logarithmic UV divergences at:

protected F-terms:

- One loop in D=8 R^4
- Two loops in D=7 $\partial^4 R^4$
- Three loops in D=6 $\partial^6 R^4$
- Four loops in D=11/2 $\partial^8 R^4$

$D = 4 + 6/L$ (two to four loops)

- Five loops in D=??? $\partial^{??} R^4$

These can be determined by duality.

(i) D=8 R^4

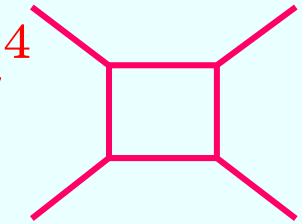
poles in ε cancel

$$\mathcal{E}^{(8)}_{(0,0)} = \lim_{\epsilon\to 0}\left(\mathbf{E}^{SL(3)}_{[10];\frac{3}{2}+\epsilon} + 2\mathbf{E}^{SL(2)}_{1-2\epsilon}\right) = \hat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\hat{\mathbf{E}}^{SL(2)}_{1}$$

- Poles in ε cancel (UV finiteness of string theory)
- Completely determined by finite expression in D=7
- String perturbation theory (in **Einstein** frame)

$$\int_{P(\alpha_1)} \mathcal{E}^{(8)}_{(0,0)} = \frac{2\zeta(3)}{y_8} + 2(\hat{\mathbf{E}}_1(T) + \hat{\mathbf{E}}_1(U)) + \frac{2\pi^2}{3}\log y_8$$

(moduli T, U, y_8)

Recall maximal SUGRA R^4  1-loop logarithm

$\log y_8$ must come from $\log(s\,l_s^2)$ in the string frame

$$\log(s\,l_s^2) = \log(s\,l_D^2) - \frac{1}{D-2}\log y_D$$

Weyl dilaton factor $l_s^{D-2} = l_D^{D-2}\,y_D^{-1}$

Coefficient is determined by unitarity and must be the same as the coefficient of $c\,\log(s/\Lambda^2)$ in supergravity.

Ultraviolet cutoff in supergravity

The coefficient of $\log y_8$ in $\mathcal{E}^{(8)}_{(0,0)}$ determines the UV logarithm in one-loop maximal supergravity in D=8 R^4

(ii) D=7 $\partial^4 R^4$

$$\mathcal{E}^{(7)}_{(1,0)} \quad = \quad \frac{1}{2}\hat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{3}{\pi^2}\hat{\mathbf{E}}^{SL(5)}_{[0010]1}$$

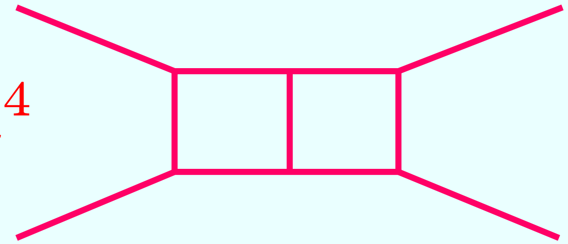
poles cancel again

- String perturbation theory

$$\int_{P(\alpha_1)} \mathcal{E}^{(7)}_{(1,0)} = \frac{\zeta(5)}{y_7^2} + \frac{1}{y_7}(\dots) + (\dots) + \frac{8\pi^2}{15}\log y_7$$

tree 1-loop 2-loop 2-loop logarithm

Agrees with two-loop supergravity UV divergence in D=7

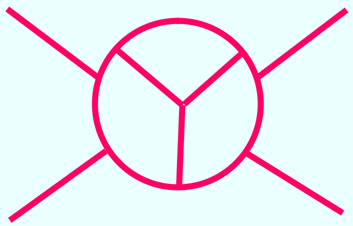
$S^2\,R^4$  + ...

Bern, Dixon, Dunbar, Perelstein, Rozowsky,1998

(iii) D=6 $\partial^6 R^4$

$$\int_{P(\alpha_1)} \mathcal{E}^{(6)}_{(0,1)} = \underbrace{\frac{2\zeta(3)^2}{3\,y_6^3}}_{\text{tree}} + \underbrace{\frac{1}{y_6^2}(\ldots)}_{\text{1-loop}} + \underbrace{\frac{1}{y_6}(\ldots)}_{\text{2-loop}} + \underbrace{(\ldots)}_{\text{3-loop}} + \underbrace{15\zeta(3)\,\log y_6}_{\text{3-loop logarithm}} + n.p.$$

Agrees with three-loop supergravity UV divergence in D=8 *

$S^3\,R^4$  + ...

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007

* Factor of 6 missing??

(iii) D=8 $\partial^6 R^4$

one-loop subdivergence
R4 coeff.

two-loop

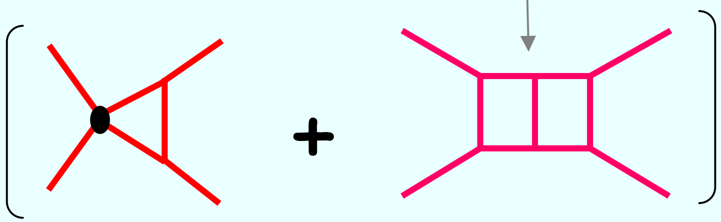
$$\mathcal{E}^{(8)}_{(0,1)} = \cdots + \frac{\pi}{9}\left(\frac{\pi}{6} + \mathbf{E}^{pert}_{(0,0)}\right)\log y_8 - \frac{\pi^2}{27}\log^2 y_8\bigg) + n.p.$$

Agrees with D=8 two-loop supergravity calculations

Sum of double pole and single pole –
includes effects of counterterm diagram

ε double pole

$S^3\, R^4$



BUT

Supergravity limit of string theory

Ooguri, MBG, Schwarz 2006

MBG, Russo, Vanhove 2010

Whether or not there are ultraviolet divergences, supergravity does not decouple from string theory.

The supergravity limit:

Fixed $\quad \ell_D^{D-2} = \ell_s^{D-2}\, y_D \quad \ell_s \to 0$ (decouples string excitations)

i.e. $\quad y_D \to \infty$

Infinite towers of charged BPS black hole states/instantons become massless and/or instantons have zero action.

Wrapped p-branes; KK charges; KK monopoles.

Supergravity limit of string theory

Explicitly: e.g. D=8 R^4

String perturbation limit: $y_8 \to 0$

$$\mathcal{E}^{(8)}_{(0,0)} = \frac{2\zeta(3)}{y_8} + 2(\hat{\mathbf{E}}_1(T) + \hat{\mathbf{E}}_1(U)) + \frac{2\pi}{3}\log y_8 + O(e^{-(y_8 T_2)^{-\frac{1}{2}}}, e^{-(y_8/T_2)^{-\frac{1}{2}}})$$

tree

one loop

D-instantons

Wrapped Dp-string World-sheets

action$\to 0$

Supergravity limit $y_8 \to \infty$

Perturbative terms swamped by condensate of zero action instantons - Poisson resum. gives

$$\mathcal{E}^{(8)}_{(0,0)} = y_8^{\frac{1}{2}}\,\mathbf{E}_{\frac{3}{2}}(T) - \frac{4\pi}{3}\log y_8 + O(e^{-\sqrt{y_8 T_2}}, e^{-\sqrt{y_8/T_2}})\,.$$

Comments on $\partial^8\mathcal{R}^4$:

- Four-loop supergravity UV divergence in $D = \frac{11}{2}$

Bern, Carrasco, Dixon, Johansson, Roiban 2009

Is there a FIVE-LOOP contribution ??

- Indications of genus $h = 5$ contribution to $\partial^8\mathcal{R}^4$

MBG, Russo, Vanhove 2010

Duality argument — MBG, Russo, Vanhove 2008

Pure spinor formalism — Berkovits, MBG, Russo, Vanhove 2009; Bjornsson, MBG 2010

- Suggests $\partial^8\mathcal{R}^4$ is a "D-term" — contributions from all h

Would lead to **SEVEN**-LOOP UV divergence in N=8 SUGRA $D = 4$

(also suggested by certain superspace arguments)

(NOTE: If $h \leq k$ for all k, UV divergences absent for $D < 4 + \dfrac{6}{h}$)

III) Evidence for 5-loop contribution to $\partial^8\mathcal{R}^4$

Jonas Bjornsson, MBG "5 loops in 24/5 dimensions", arXiv:1004.2692

Based on a pure spinor formalism for the superparticle.

World-line formalism with coordinates:

Bosons: World-line scalars $X^m, \lambda^\alpha, \bar\lambda_\alpha$

16, 11, 11 no. of zero modes

World-line vectors $P_m, w_\alpha, \bar w^\alpha$

16L, 11L, 11L L= loop number

Fermions: World-line scalars θ^α, r_α

16, 11

World-line vectors d_α, s^α

16L, 11L

pure spinors $\lambda\gamma^m\lambda = 0$, $\lambda\gamma^m r = 0$ $\quad \lambda, \bar\lambda, w, \bar w, r, s$

For supergravity these are doubled $\hat\theta, \hat\lambda, \hat{\bar\lambda}, \hat w, \hat{\bar w}, \hat r, \hat s$

BRST operator $Q_{tot} = Q + \hat{Q}$

$$Q = \lambda d + \bar{w} r \,, \qquad \hat{Q} = \hat{d}\hat{\lambda} + \hat{r}\hat{\bar{w}}$$

Composite $b, \hat{b}$ ghost – (3L-3) insertions in L-loop amplitude

$$b = -\frac{1}{4}\left(\frac{P^m \bar{\lambda}\,\gamma_m d}{(\lambda\bar{\lambda})} + \frac{\bar{\lambda}\,\gamma_{mnp} r\,[(d\gamma^{mnp} d)}{384\,(\lambda\bar{\lambda})^2} + \ldots\right),$$

so that

$$[Q_{tot}, (b + \hat{b})] = H \,, \qquad [Q_{tot}, (b - \hat{b})] = 0$$

Integrate vertex insertions $V(k_r, \tau_r)$ (functions of $(d\,, \theta)$)

For supergravity:

$$V \sim P^m\, P^m\, G_{mn} + d^\alpha\, \hat{d}^\beta W_{\alpha\beta} + \ldots$$

superfields $G_{mn}(X, \theta)\,, W_{\alpha\beta}(X, \theta)$

L-loop amplitude :

$$\int \mathcal{D}\Phi\mathcal{D}\hat{\Phi} \left(\mathcal{N}\hat{\mathcal{N}} \prod_{i=1}^{3(L-1)} \left(\int_0^{T_i} \frac{d\tau}{T_i} b \int_0^{T_i} \frac{d\tau}{T_i} \hat{b} \right) \int \prod_{r=1}^{4} d\tau_r \, V(k_1, \tau_1) \dots V(k_4, \tau_4) \, e^{-S} \right)$$

Regulator for large $\lambda, \bar{\lambda}$ divergences $\mathcal{N} = e^{-\lambda\bar{\lambda}+\theta r-\lambda\, d\, s+\dots}$

(3L-3) $b, \hat{b}$ insertions in L-loop amplitude

Saturation of fermionic zero modes. Requires detailed consideration of modes coming from regulator $\mathcal{N}\hat{\mathcal{N}}$, b insertions and vertex insertions, V.

Constrains pattern of diagrams that contribute to amplitude.

Consider saturation of the 16 L fermionic d zero modes.

11 L of these are soaked up by the d′s in the factor $(s\ d)^{11L}$ leaving 5L d zero modes to be soaked up by b insertions and vertex operators.

Period matrix $${}_{IJ} = \oint_{B_I} \omega_J$$

1-form $\omega_I = a_I^i \, d\tau_i$ with $a_I^i = \pm 1$

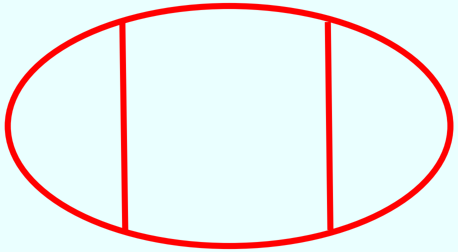
B-cycles – basis of L loops

Term in b zero mode with most d zero modes

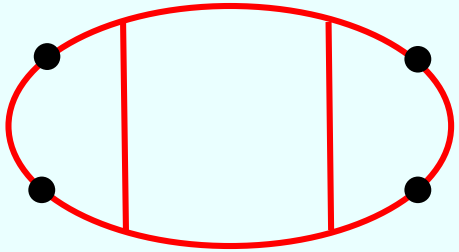
$$\int_0^{T_i} \frac{d\tau}{T_i} \; b\,|_{zero} = d_\gamma^I \, d_\delta^J \frac{\partial \;\; {}_{IJ}}{\partial T_i} \qquad \gamma\,,\ \delta = 1, \ldots, 5$$

Furthermore, $d_\alpha^2 = 0$ so each d component occurs only once

Three loops



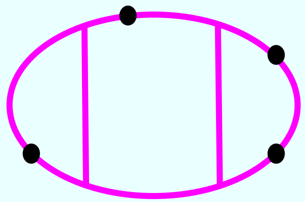
$\partial^8\,\mathcal{R}^4\,\Lambda^{3D-20}$



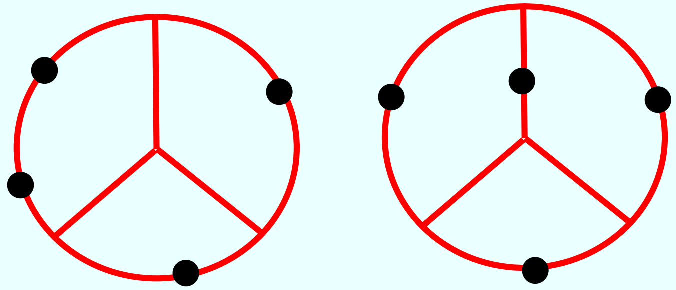
Ladder amplitude

The three-loop skeletons

Not allowed

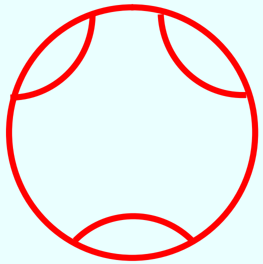


$\partial^6\,\mathcal{R}^4\,\Lambda^{3(D-6)}$

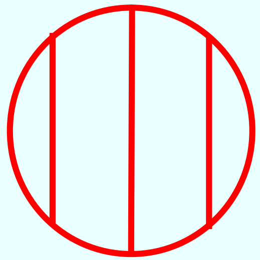


Two leading contributions

The five four-loop skeletons
to which vertices must be inserted



1



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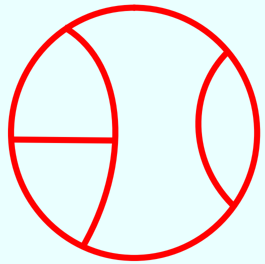

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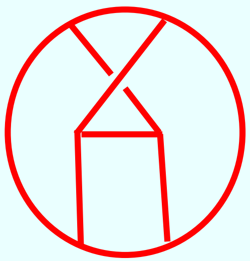

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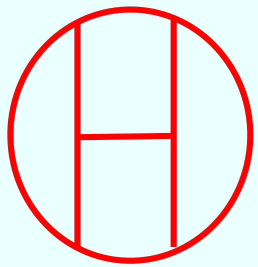
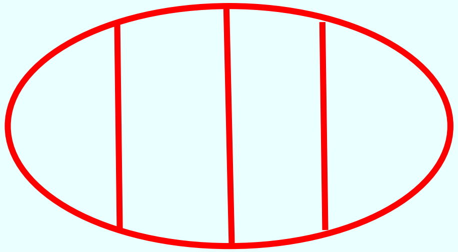
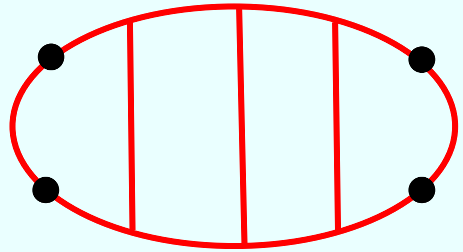


5

Four loops

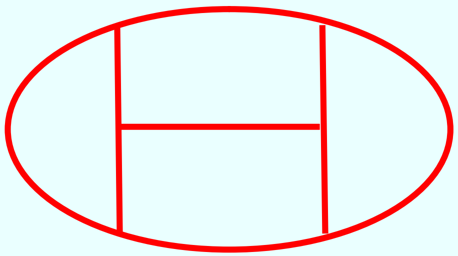


$\partial^{12}\,\mathcal{R}^4\,\Lambda^{4D-26}$

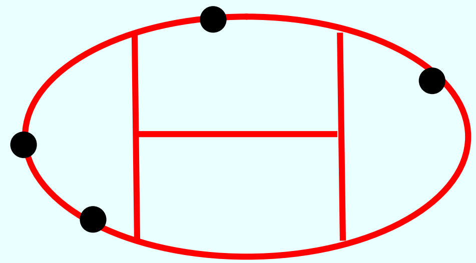


Ladder amplitude

Two of the five
four-loop skeletons



$\partial^{8}\,\mathcal{R}^4\,\Lambda^{4D-22}$



Example of a leading
amplitude

Five loops

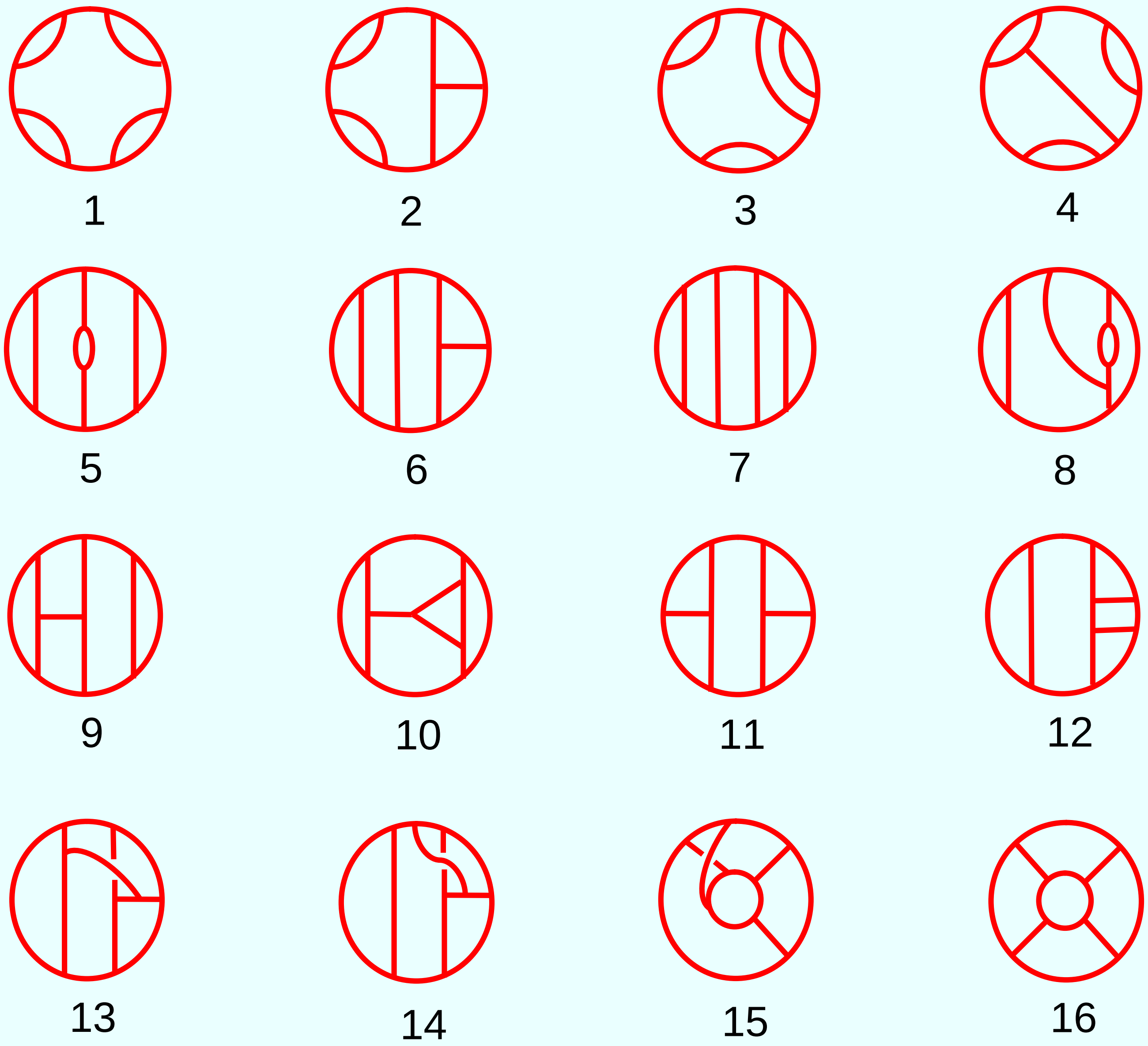
New feature arises at five loops. Require 3L-3 = 12 b insertions. Can now get power

$$\left(\frac{r}{\lambda\bar{\lambda}}\right)^{12}$$

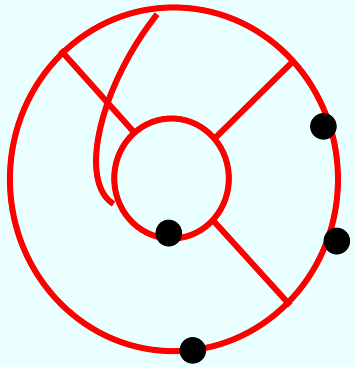
Small- $\lambda\,,\bar{\lambda}$ singularity and more than 11 r's, giving 0/0. Needs new regulator (Berkovits-Nekrasov) changes systematics.

Five-loop skeletons

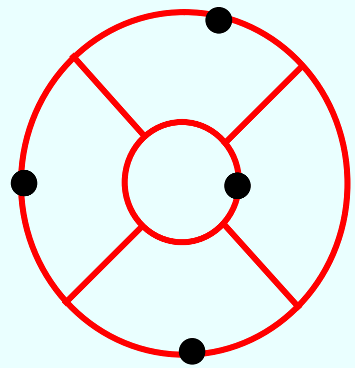
to which vertices must be inserted



Last two diagrams give leading contribution:



$$\partial^8 \mathcal{R}^4 \Lambda^{5D-24}$$



Nonplanar

Planar

Leading contributions obtained by attaching
4 vertices of the form $P^m \, P^n \, G_{mn}$

Gives four cancelled propagators, leaving the vacuum
(skeleton) diagrams

$$\int d^{5D} k \, (k^2)^{-12} \sim \Lambda^{5D-24}$$

Log UV divergence in D=24/5 dimensions proportional to

$$\partial^8\,\mathcal{R}^4\log\Lambda$$

Furthermore, general arguments suggest higher loops also contribute to $\partial^8\,\mathcal{R}^4$ so this interaction is not protected.

c.f. If $\partial^8\,\mathcal{R}^4$ were protected the low energy five-loop ampltude would have behaved as

$$\partial^{10}\,\mathcal{R}^4\,\Lambda^{5D-26}$$

consistent with $D = 4 + \frac{6}{L}$ and UV finiteness of N=8 supergravity in D=4

Strongly suggests that supersymmetry protects interactions of the form $\partial^{2k}\,\mathcal{R}^4$ up to k=3. The interaction $\partial^8\,\mathcal{R}^4$ is unprotected and likely to have an ultraviolet divergence at seven loops in D=4 dimensions.