Properties of low energy graviton scattering amplitudes

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"String Theory: Formal Developments and Applications"
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I) Non-perturbative aspects of maximally supersymmetric closed string theory scattering amplitudes.
Low energy expansion in flat space in $D < 11$ dimensions.
Role of duality symmetries.

II) Relationship to multiloop supergravity quantum field theory.
Ultraviolet divergences in $D$ dimensions.

III) Perturbative supergravity using pure spinor quantum mechanics.
Explicit evidence for UV divergence of form $d^4R^4$ in $D=4$ maximal supergravity.
Non-perturbative properties of four-point scattering in Type II string theory:


Related work by: Pioline, Obers, Kiritsis, Basu, Sethi, ...

Five-loop maximal supergravity and evidence for a seven-loop UV divergence in $\text{N}=8$ supergravity

4) Jonas Bjornsson, MBG “5 loops in 24/5 dimensions”, arXiv:1004.2692
I) **Dualities of type II theory**

on \( d=(10-D) \)-torus \( \mathcal{T}^d \)

Duality invariance implies relations between perturbative and nonperturbative terms in the \( S \)-matrix.

Non-trivial dependence on the moduli (scalar fields),
- in contrast to classical supergravity

**Moduli space**: space of scalar fields

\[
G(\mathbb{Z}) \backslash G(\mathbb{R}) / K
\]

- discrete identification-breaks symmetry to discrete subgroup
- maximal compact subgroup

Scalar fields in maximally supersymmetric supergravity

\(e.g.\) Type IIB theory in \( D=10 \) dimensions

\[
SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / O(2)
\]
U-duality groups for type II in $D$ dimensions on $T^d$, $D = 10 - d$

<table>
<thead>
<tr>
<th>Dimension $D$</th>
<th>$G_d(\mathbb{Z}) = E_{d+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10A</td>
<td>1</td>
</tr>
<tr>
<td>10B</td>
<td>$SL(2, \mathbb{Z})$</td>
</tr>
<tr>
<td>9</td>
<td>$SL(2, \mathbb{Z})$</td>
</tr>
<tr>
<td>8</td>
<td>$SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$</td>
</tr>
<tr>
<td>7</td>
<td>$SL(5, \mathbb{Z})$</td>
</tr>
<tr>
<td>6</td>
<td>$SO(5, 5, \mathbb{Z})$</td>
</tr>
<tr>
<td>5</td>
<td>$E_{6(6)}(\mathbb{Z})$</td>
</tr>
<tr>
<td>4</td>
<td>$E_{7(7)}(\mathbb{Z})$</td>
</tr>
<tr>
<td>3</td>
<td>$E_{8(8)}(\mathbb{Z})$</td>
</tr>
</tbody>
</table>

decompactification
Four (super)-Graviton Scattering in type II

\[ A_D(s, t, u) = A_D^{analytic}(s, t, u) + A_D^{nonan}(s, t, u) \]

- local term in eff. action contains massless thresholds
- nonlocal terms in eff. action (IR divergent for \( D \leq 4 \)).

\[ A_D^{analytic}(s, t, u) = \mathcal{R}^4 T_D(s, t, u) \]

\[ s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_4)^2, \quad u = -(k_1 + k_3)^2 \]

\[ \mathcal{R} \quad \text{Linearised supercurvature – describes 256 physical states in supermultiplet.} \]

Momentum expansion:

\[ T_D(s, t, u) = \sum_{p, q} \mathcal{E}^{(D)}_{(p, q)} \sigma_2^p \sigma_3^q \]

- Power series in
  \[ \sigma_2 = s^2 + t^2 + u^2 \]
  \[ \sigma_3 = s^3 + t^3 + u^3 \]

Coefficients are duality invariants functions of moduli
Duality - invariant effective IIB action

(Einstein frame) Einstein-Hilbert Higher-derivative interactions

\[ S = \frac{1}{\ell_{D-2}^D} \int d^D x \sqrt{-G(D)} \, R + S_{\text{local}} + \ldots \]

\[ S_{\text{local}}^{D} = \ell_{D}^{8-D} \int d^D x \sqrt{-G(D)} \left( \mathcal{E}_{(0,0)}^{(D)} \mathcal{R}^4 + \ell_{D}^{4} \mathcal{E}_{(1,0)}^{(D)} \partial^4 \mathcal{R}^4 + \ell_{D}^{6} \mathcal{E}_{(0,1)}^{(D)} \partial^6 \mathcal{R}^4 + \ldots \right) \]

Planck length
\[ \ell_{D}^{D-2} = \ell_{D}^{D-1} \frac{1}{r_d} \]

String length scale
\[ \ell_{s}^{4} = \ell_{10}^{4} g_{s}^{-1} = \ell_{10} e^{-\phi} = \ell_{10} \tau_2 \]

(complex coupling \( \tau = \tau_1 + i\tau_2 \))

\( \mathcal{E}_{(p,q)}^{(D)} \) are \( E_{d+1} \) - invariant coefficients functions of moduli

How may these be determined??

1/2 BPS 1/4 BPS 1/8 BPS
Simple cases can be determined by supersymmetry:

Sketch:

Low energy expansion of amplitude:

\[ \ell_D^{-2} S = S^{(0)} + \ell_D^6 S^{(6)} + \ell_D^{10} S^{(10)} + \ell_D^{12} S^{(12)} + \ldots \]

Classical action

Expand SUSY transformation on fields \( \Phi \):

\[ \delta \Phi = (\delta^{(0)} + \ell_D^6 \delta^{(6)} + \ell_D^{10} \delta^{(10)} + \ell_D^{12} \delta^{(12)} + \ldots) \Phi \]

SUSY requires:

\[ \delta S = (\delta^{(0)} + \ell_D^6 \delta^{(6)} + \ldots) (S^{(0)} + \ell_D^6 S^{(6)} + \ldots) = 0 \]

Closure of superalgebra:

\[ [\delta, \delta] \Phi = [\delta^{(0)} + \ell_D^6 \delta^{(6)} + \ldots, \delta^{(0)} + \ell_D^6 \delta^{(6)} + \ldots] \Phi \]

\[ = a \cdot \partial \Phi + \text{eqs. of motion} \]
e.g. $D=10$ IIB

duality group $SL(2, \mathbb{Z})$

modulus $\tau = \tau_1 + i \tau_2$

$\tau_2 \equiv e^{-\phi} = g_B^{-1}$

coupling

i) For $E^{(10)}_{(0,0)} \mathcal{R}^4$

$E^{(10)}_{(1,0)} \sigma_2 \mathcal{R}^4$

$S^{(6)}$

$S^{(10)}$

SUSY implies Laplace eigenvalue equation

$$\Delta^{(10)} = \tau_2^2 \partial_\tau \partial_{\bar{\tau}} \quad \rightarrow \quad \Delta E^{(10)}_{(p,0)} = s(s - 1) E^{(10)}_{(p,0)}$$

$s = p + \frac{3}{2}$

Invariant laplacian for $SL(2, \mathbb{R})/SO(2)$

$s = \frac{3}{2}$ for $\mathcal{R}^4$

$s = \frac{5}{2}$ for $\sigma_2 \mathcal{R}^4$

Unique solution is nonholomorphic Eisenstein series

subject to moderate growth b.c.

(perturbation theory for large $\tau_2$)

$$E^{(10)}_{(p,0)} = E_s = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}}$$
\[
E_s = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}}
\]

two power behaved terms

\[\sim 2\zeta(2s)\tau_2^s + (\ldots)\zeta(2s - 1)\tau_2^{1-s} + \sum_{k\neq0} \mu(k, s) (e^{2\pi ik\tau} + c.c.) (1 + O(\tau_2^{-1}))\]

- **TREE-level term**
  \((\tau_2 = g_s^{-1})\)

- **GENUS-(s - \frac{1}{2}) term**

- **D-INSTANTON terms**
  Infinite no. of pert. Corrections-
  non-zero Fourier modes.
  Interesting measure

\[\int d\tau_1\] projects onto "Constant Term" - zero Fourier mode - power behaved terms, kills instantons

- **NO HIGHER LOOP** perturbative terms
- **Non-renormalization** at higher loops

**examples:**
\[E_{\frac{3}{2}} R^4, \quad E_{\frac{5}{2}} s^2 R^4\]

- tree-level + one-loop
- tree-level + two-loop
Motivated by four-graviton scattering amplitude, and consistent with supersymmetry expectations (but not yet Derived from SUSY).

Inhomogeneous Laplace eigenvalue equation

$$(\Delta_\tau - 12) \mathcal{E}^{(10)}_{(0,1)} = -E_{3\frac{3}{2}}^2 E_{3\frac{3}{2}}^2 = -\left(\mathcal{E}^{(10)}_{(0,0)}\right)^2$$

Motivated by four-graviton scattering amplitude, and consistent with supersymmetry expectations (but not yet Derived from SUSY).

$$\delta S = \delta^{(0)} S^{(12)} + \delta^{(12)} S^{(0)} + \delta^{(6)} S^{(6)} + \ldots$$

Constant term (zero Fourier mode) contains:

Genus 0, 1, 2, 3 + instanton anti-instanton pairs
Note:

Non-renormalisation. Interactions of the form $\partial^{2k} \mathcal{R}^4$ have contributions for genus-$h$ with $h \leq k$ (for $h > 1$)

Does this continue to hold for $k > 3$??

[comments later]
Eisenstein series for higher-rank duality groups

General Eisenstein series depends on \( r = \text{rank } G \) parameters \( s_i \quad i = 1, \ldots, r \) (Selberg, Langlands)

But Eisenstein series for a maximal parabolic subgroup, \( P(\beta) \), defined with respect to a simple root \( \beta \), depends on only one non-zero parameter

\[
s = s_\beta \neq 0 \quad s_i = 0, \ (i \neq \beta)
\]

\[
E^G_\beta(g; \mu) = \sum_{\gamma=P(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{<\mu, H(\gamma g)>}
\]

\( P(\mathbb{Z}) = P(\mathbb{R}) \cap G(\mathbb{Z}) \)

parabolic subgroup

\( H(g) \) Cartan subalgebra

Automorphise – sum over coset

\( \mu = (\mu_1, \ldots, \mu_r) \) Vector parameterising Cartan torus
e.g. $GL(d)$ Parabolic subgroup of matrices of form:

$$P(n_1, \ldots, n_q) = \begin{pmatrix} U & * \\ 0 & a \end{pmatrix}$$

$d \times d$ matrix $U \in GL(d - 1)$ is associated with Dynkin label $[0 \cdots 01]$.

**NOTATION:**

$E_{[0 \cdots 010 \cdots 0];s}^G = \text{Maximal Parabolic Eisenstein series for Parabolic subgroup } P(\beta) \text{ associated with Dynkin label } [0 \cdots 010 \cdots 0]$.

[Standard $SL(2, \mathbb{Z})$ Eisenstein series: $E_s = E^{SL(2)}_{[1];s}$]
e.g. \( \text{SL}(2,\mathbb{Z}) \)

\[
E_s^{\text{SL}(2)} = \sum_{\gamma \in \text{N}(\mathbb{Z}) \setminus \text{SL}(2,\mathbb{Z})} \text{Im}(\gamma(\tau))^s
\]

\[
N = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}
\]

\[
\gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1
\]
Epstein Series
e.g. $SL(d,\mathbb{Z})$:

$$E_{[10\ldots0];s}^{SL(d)} = \sum_{m \in \mathbb{Z}^d} \frac{1}{(m^i G_{ij} m^j)^s}$$

dependence on $SL(d)/SO(d)$ moduli

$$E_{[01\ldots0];s}^{SL(d)} = \sum_{m, n \in \mathbb{Z}^d} \frac{1}{(m^i n_j G_{ik} G_{j\ell} m^{[k} n^{l]})^s}$$

antisymmetrised sums

Lattice sums are much more subtle for other duality groups $SO(5,5), E_6, E_7, E_8$

Evaluate sums by “brute force” (Mathematica)
These series satisfy Laplace equation - quadratic Casimir

\[
(\Delta_{G/K} - \lambda) E_{0\ldots010\ldots0}^G; s = 0
\]

eigenvalue depends on \( G, \beta, s \)
Consistency conditions determine coefficients:

Laplace equations + boundary conditions
Laplace equations in various dimensions:

Decompactification of laplacian starting from $D$ and using $D=10$ expressions as boundary conditions.

$$\Delta^{(D)} = \Delta_{E11-D/K_D}$$

$$\begin{align*}
R^4 & \quad \left( \Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2} \right) \epsilon^{(D)}_{(0,0)} = 6\pi \delta_{D-8,0} \\
\partial^4 R^4 & \quad \left( \Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2} \right) \epsilon^{(D)}_{(1,0)} = 40\zeta(2) \delta_{D-7,0} \\
\partial^6 R^4 & \quad \left( \Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2} \right) \epsilon^{(D)}_{(0,1)} = -\left( \epsilon^{(D)}_{(0,0)} \right)^2 + 120\zeta(3) \delta_{D-6,0}
\end{align*}$$

Singularity in $D=8$

Singularity in $D=7$

Singularity in $D=6$
Consistent degeneration limits of torus:

(i) Decompactify from $D$ to $D+1$ dimensions

\[ E_{d+1} \to E_d \quad (d = 10 - D) \quad r_d \to \infty \]

Power-behaved terms in \( \int_{P(\alpha_{d+1})} \mathcal{E}^{(D)}_{(p,q)} \)

zero Fourier mode - kills instantons

“Constant term in parabolic subgroup” \( P(\alpha_{d+1}) \)
(i) Decompactification of circle radius $r_d$

Consistency conditions:

Terms proportional to $r_d$ survive in $D+1$

$$R^4 \left( \frac{\ell_{D+1}}{l_D} \right)^{8-D} \int_{P(\alpha_d)} \mathcal{E}^{(D)}_{(0,0)} = \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(0,0)} + \left( \frac{r_d}{l_{D+1}} \right)^{8-D}$$

$$\partial^4 R^4 \left( \frac{\ell_{D+1}}{l_D} \right)^{12-D} \int_{P(\alpha_d)} \mathcal{E}^{(D)}_{(1,0)} = \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(1,0)} + \left( \frac{r_d}{l_{D+1}} \right)^{6-D} \mathcal{E}^{(D+1)}_{(0,0)} + \left( \frac{r_d}{l_{D+1}} \right)^{12-D}$$

$$\partial^6 R^4 \left( \frac{\ell_{D+1}}{l_D} \right)^{14-D} \int_{P(\alpha_d)} \mathcal{E}^{(D)}_{(0,1)} = \frac{r_d}{\ell_{D+1}} \mathcal{E}^{(D+1)}_{(0,1)} + \left( \frac{r_d}{l_{D+1}} \right)^{4-D} \mathcal{E}^{(D+1)}_{(1,0)} + \left( \frac{r_d}{l_{D+1}} \right)^{14-D} \mathcal{E}^{(D+1)}_{(0,0)}$$

- Limited set of powers of $r$
- Note that all the $D+1$ coefficients are contained in $\partial^6 R^4$
String perturbation expansion

\[ \frac{1}{g_s^2} V_d = \frac{1}{y_D} \to \infty \quad \text{fixed} \quad \frac{r_d}{\ell_s} \]

\[ E_{d+1} \to SO(d, d) \quad \text{T-duality} \]

![Diagram with labels and points]

Perturbative terms in \[ \int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(p,q)} \]

Powers of \( y_D \) correspond to world-sheet genus

Constant term in parabolic subgroup \( P(\alpha_1) \)
(ii) String perturbation expansion

Powers of $y_D$ correspond to world-sheet genus

### genus 0

\[
R^4 \quad \int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(0,0)} = \frac{l_s^{8-D}}{l_D^{8-D}} \left( \frac{2\zeta(3)}{y_D} + I_{(0,0)}^{(1)} \right)
\]

### genus 1

\[
\partial^4 R^4 \quad \int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(1,0)} = \frac{l_s^{12-D}}{l_D^{12-D}} \left( \frac{2\zeta(4)}{y_D} + I_{(1,0)}^{(1)} + y_D I_{(1,0)}^{(2)} \right)
\]

### genus 2

\[
\partial^6 R^4 \quad \int_{P(\alpha_1)} \mathcal{E}^{(D)}_{(0,1)} = \frac{l_s^{14-D}}{l_D^{14-D}} \left( \frac{2\zeta^2(3)}{3y_D} + I_{(0,1)}^{(1)} + y_D I_{(0,1)}^{(2)} + y_D^2 I_{(0,1)}^{(3)} + O(e^{-4/y_D}) \right)
\]
(iii) 11-dimensional supergravity limit \( \nu_{d+1} \to \infty \)
vol. of M-theory torus

Large M-theory torus matches Feynman diagrams of eleven-dimensional supergravity one, two loops

\[ E_{d+1} \to SL(d + 1) \]
geometric symmetry

Constant term in parabolic subgroup \( P(\alpha_2) \)

Perturbative terms in \( \int_{P(\alpha_2)} \mathcal{E}^{(D)}_{(p,q)} \)
SOLUTIONS to Laplace equations and b.c.’s:

\[ R^4 \]

\[ \mathcal{E}^{(3)}_{(0,0)} = E_{E_8}^{E_8 [10000000]; \frac{3}{2}} \]

\[ \partial^4 R^4 \]

\[ \mathcal{E}^{(3)}_{(0,0)} = E_{E_8}^{E_8 [10000000]; \frac{5}{2}} \]

More complicated for

\[ \mathcal{E}^{(3)}_{(0,1)} \partial^6 R^4 \]

Perturbative expansions match in all three limits.

How can these expressions be interpreted in terms of sums over 1/2-BPS, 1/4-BPS, 1/8-BPS states ??

MBG, Miller, Russo, Vanhove 2010

(Pioline 2010)
Solutions in all dimensions for $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$

<table>
<thead>
<tr>
<th>$G_d = E_{d+1(d+1)}$</th>
<th>$\mathcal{R}^4$</th>
<th>$\partial^4\mathcal{R}^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_8(8)(\mathbb{Z})$</td>
<td>$E_8^{\mathbb{E}}_{[10000000]; \frac{3}{2}}$</td>
<td>$\frac{1}{2} E_8^{\mathbb{E}}_{[10000000]; \frac{5}{2}}$</td>
</tr>
</tbody>
</table>
(i) Decompactification of circle radius $r$

from $D=3$ to $D=4$

\[ R^4 \quad E^{E_8}_{[10000000];\frac{3}{2}} = r^6 E^{E_7}_{[1000000];\frac{3}{2}} + r^{10} \frac{3\zeta(5)}{\pi} \]

+ instantons

$R^4$ coefficient in $D=4$

\[ \partial^4 R^4 \quad E^{E_8}_{[10000000];\frac{5}{2}} = r^{10} \frac{1}{2} E^{E_7}_{[1000000];\frac{5}{2}} + r^{12} \frac{\zeta(3)}{\pi} E^{E_7}_{[1000000];\frac{3}{2}} + r^{18} \frac{\pi^2 \zeta(9)}{15} \]

$\partial^4 R^4$ coefficient in $D=4$

Note: Many terms have to vanish for this to work!!

\[ E^{E_8}_{[10000000];s} = 756r^{36} + 126r^{92-8s} + 576r^{60-4s} + 126r^{8s} + 576r^{14+4s} \]

(suppressing all coefficient $E_7$ Eisenstein series for simplicity)
(ii) $D=3$ String Perturbation Theory

\[
\text{e.g., } D=3 \ (d=7) \quad \text{String coupling, } r^{-4} = y_3 = \frac{g_s^2}{V_7}
\]

\[
\begin{align*}
R^4 & \quad \mathcal{E}^E_{E_8}^{E_8} \left[10000000; \frac{3}{2}\right] = r^{24} 2\zeta(3) + r^{20} \frac{3}{2\pi} \mathcal{E}^{SO(7,7)}_{[10000000]; \frac{5}{2}} + \text{instantons} \\
\partial^4 R^4 & \quad \mathcal{E}^E_{E_8}^{E_8} \left[10000000; \frac{5}{2}\right] = r^{40} \zeta(5) + \frac{7}{24\pi} r^{36} \mathcal{E}^{SO(7,7)}_{[10000000]; \frac{9}{2}} + \frac{2}{3} r^{32} \mathcal{E}^{SO(7,7)}_{[0000010]; 2}
\end{align*}
\]

Verify by explicit genus-one and genus-two calculations

Note nonrenormalisation conditions
(iii) 11-dimensional supergravity limit $\mathcal{V}_{d+1} \to \infty$

e.g., $D=3$ ($d=7$) $r^{1+d} = \mathcal{V}_{d+1}$ vol. of $M$-theory torus

$R^4 \quad \mathbb{E}^{E_8}_{[10000000]; \frac{3}{2}} = r^{32} 4\zeta(2) + r^{30} \mathbb{E}^{SL(8)}_{[1000000]; \frac{3}{2}}$ + instantons

$\partial^4 R^4 \quad \mathbb{E}^{E_8}_{[10000000]; \frac{5}{2}} = \frac{1}{2} r^{50} \mathbb{E}^{SL(8)}_{[1000000]; \frac{5}{2}} + \frac{2\zeta(3)}{\pi^2} r^{48} \mathbb{E}^{SL(8)}_{[0100000]; 2} + \frac{4\zeta(4)}{3} r^{54} \mathbb{E}^{SL(8)}_{[10000000]; -\frac{1}{2}}$

Precisely reproduced by one-loop and two-loop supergravity in eleven dimensions on $(d+1)$-torus.
very
NOTE: Many terms have to vanish for this to work!!

For generic $s$

$$E^{E_{8}}_{[10000000];s} \sim 64r^{6s+1} + 448r^{41-2s} + 280r^{4(s+2)} + 448r^{2(s+9)}$$
$$+ r^{92-8s} + r^{8s} + 64r^{70-6s} + 280r^{54-4s} + 14r^{34} + 560r^{28}$$

(setting all coefficient SL(8) Eisenstein series = 1 for simplicity)

All but 2 or 3 vanish for $s=3/2$ or $5/2$
Determining Supergravity UV divergences

“Critical” Cases

Maximal SUGRA has logarithmic UV divergences at:

<table>
<thead>
<tr>
<th>Order of Loops</th>
<th>Dimension</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>One loop</td>
<td>D=8</td>
<td>$R^4$</td>
</tr>
<tr>
<td>Two loops</td>
<td>D=7</td>
<td>$\partial^4 R^4$</td>
</tr>
<tr>
<td>Three loops</td>
<td>D=6</td>
<td>$\partial^6 R^4$</td>
</tr>
<tr>
<td>Four loops</td>
<td>D=11/2</td>
<td>$\partial^8 R^4$</td>
</tr>
<tr>
<td>Five loops</td>
<td>D=???</td>
<td>$\partial^{??} R^4$</td>
</tr>
</tbody>
</table>

$D = 4 + 6/L$

These can be determined by duality.
(i) D=8  \( R^4 \)

\[ \mathcal{E}^{(8)}_{(0,0)} = \lim_{\epsilon \to 0} \left( E^{SL(3)}_{[10]; \frac{3}{2} + \epsilon} + 2E^{SL(2)}_{1 - 2\epsilon} \right) = \hat{E}^{SL(3)}_{[10]; \frac{3}{2}} + 2\hat{E}^{SL(2)}_1 \]

- Poles in \( \epsilon \) cancel (UV finiteness of string theory)
- Completely determined by finite expression in D=7
- String perturbation theory \( \text{(in Einstein frame)} \)

\[ \int_{P(\alpha_1)} \mathcal{E}^{(8)}_{(0,0)} = \frac{2\zeta(3)}{y_8} + 2(\hat{E}_1(T) + \hat{E}_1(U)) + \frac{2\pi^2}{3} \log y_8 \]

(moduli T, U, y_8)

Recall maximal SUGRA

\( R^4 \)

1-loop logarithm
\[ \log y_8 \text{ must come from } \log(s l_s^2) \text{ in the string frame} \]

\[ \log(s l_s^2) = \log(s l_D^2) - \frac{1}{D-2} \log y_D \]

Weyl dilaton factor \( l_s^{D-2} = l_D^{D-2} y_D^{-1} \)

Coefficient is determined by unitarity and must be the same as the coefficient of \( c \log(s/\Lambda^2) \) in supergravity.

Ultraviolet cutoff in supergravity

The coefficient of \( \log y_8 \) in \( \mathcal{E}^{(8)}_{(0,0)} \) determines the UV logarithm in one-loop maximal supergravity in \( D=8 \) \( \mathbb{R}^4 \).
(ii) D=7 \[ \partial^4 R^4 \]

\[ \mathcal{E}_{(1,0)}^{(7)} = \frac{1}{2} \hat{E}_{[1000]; \frac{5}{2}}^{SL(5)} + \frac{3}{\pi^2} \hat{E}_{[0010]}^{SL(5)} \]

poles cancel again

- String perturbation theory

\[
\int_{P(\alpha_1)} \mathcal{E}_{(1,0)}^{(7)} = \frac{\zeta(5)}{y_7^2} + \frac{1}{y_7} (\ldots) + (\ldots) + \frac{8\pi^2}{15} \log y_7
\]

tree 1-loop 2-loop 2-loop logarithm

Agrees with two-loop supergravity UV divergence in D=7

\[ S^2 R^4 \]

Bern, Dixon, Dunbar, Perelstein, Rozowsky, 1998
(iii) D=6 \quad \partial^6 R^4

\int_{P(\alpha_1)} \mathcal{E}^{(6)}_{(0,1)} = \frac{2\zeta(3)^2}{3y_6^3} + \frac{1}{y_6^2} (\ldots) + \frac{1}{y_6} (\ldots) + (\ldots) + 15\zeta(3) \log y_6 + n.p.

tree \quad 1\text{-loop} \quad 2\text{-loop} \quad 3\text{-loop} \quad 3\text{-loop logarithm}

Agrees with three-loop supergravity UV divergence in D=8 *

\begin{align*}
S^3 \quad R^4 & \quad + \ldots \ldots \\
\end{align*}

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007

* Factor of 6 missing??
(iii) $D=8 \quad \partial^6 R^4$  

\[ \mathcal{E}^{(8)}_{(0,1)} = \cdots + \frac{\pi}{9} \left( \frac{\pi}{6} + E^{pert}_{(0,0)} \right) \log y_8 - \frac{\pi^2}{27} \log^2 y_8 \]  

Agrees with $D=8$ two-loop supergravity calculations

Sum of double pole and single pole – includes effects of counterterm diagram

\[ S^3 R^4 \left[ \begin{array}{c} \text{double pole} \\ \varepsilon \end{array} \right] + \]
Whether or not there are ultraviolet divergences, supergravity does not decouple from string theory.

The supergravity limit:

\[ \ell_D^{D-2} = \ell_s^{D-2} y_D \quad \ell_s \to 0 \quad \text{(decouples string excitations)} \]

\[ \text{i.e.} \quad y_D \to \infty \]

Infinite towers of charged BPS black hole states/instantons become \textbf{massless} and/or instantons have \textbf{zero action}.

\textbf{Wrapped p-branes; KK charges; KK monopoles.}
Supergravity limit of string theory

Explicitly: e.g. D=8 $R^4$

String perturbation limit: $y_8 \to 0$

$$\mathcal{E}^{(8)}_{(0,0)} = \frac{2\zeta(3)}{y_8} + 2(\hat{E}_1(T) + \hat{E}_1(U)) + \frac{2\pi}{3} \log y_8 + O(e^{-(y_8 T_2)^{-\frac{1}{2}}} , e^{-(y_8 / T_2)^{-\frac{1}{2}}})$$

tree \hspace*{1cm} \text{one loop} \hspace*{1cm} \text{D-instantons} \hspace*{1cm} \text{Wrapped Dp-string World-sheets}

Supergravity limit $y_8 \to \infty$ \hspace*{1cm} action $\to 0$

Perturbative terms swamped by condensate of zero action instantons - Poisson resum. gives

$$\mathcal{E}^{(8)}_{(0,0)} = y_8^{\frac{1}{2}} \mathcal{E}_{\frac{3}{2}}(T) - \frac{4\pi}{3} \log y_8 + O(e^{-\sqrt{y_8 T_2}}, e^{-\sqrt{y_8 / T_2}}).$$
Comments on $\partial^8 R^4$:

- Four-loop supergravity UV divergence in $D = \frac{11}{2}$

  Bern, Carrasco, Dixon, Johansson, Roiban 2009

  Is there a FIVE-LOOP contribution??

- Indications of genus $h = 5$ contribution to $\partial^8 R^4$

  MBG, Russo, Vanhove 2010

  Duality argument
  Pure spinor formalism

  MBG, Russo, Vanhove 2008
  Berkovits, MBG, Russo, Vanhove 2009
  Bjornsson, MBG 2010

- Suggests $\partial^8 R^4$ is a “D-term” contributions from all $h$

  Would lead to SEVEN-LOOP UV divergence in N=8 SUGRA
  (also suggested by certain superspace arguments)

  $D = 4$

  (NOTE: If $h \leq k$ for all $k$, UV divergences absent for $D < 4 + \frac{6}{h}$)
III) Evidence for 5-loop contribution to $\partial^8 \mathcal{R}^4$

Jonas Bjornsson, MBG “5 loops in 24/5 dimensions”, arXiv:1004.2692

Based on a pure spinor formalism for the superparticle.

World-line formalism with coordinates:

Bosons: World-line scalars $X^m, \lambda^\alpha, \bar{\lambda}_\alpha$

16, 11, 11 no. of zero modes

World-line vectors $P_m, w_\alpha, \bar{w}^\alpha$

16L, 11L, 11L L = loop number

Fermions: World-line scalars $\theta^\alpha, r_\alpha$

16, 11

World-line vectors $d_\alpha, s^\alpha$

16L, 11L

pure spinors $\lambda^m \gamma^m \lambda = 0$

$\lambda, \bar{\lambda}, w, \bar{w}, r, s$

For supergravity these are doubled $\hat{\theta}, \hat{\lambda}, \hat{\bar{\lambda}}, \hat{w}, \hat{\bar{w}}, \hat{r}, \hat{s}$
BRST operator

\[ Q_{tot} = Q + \hat{Q} \]

\[ Q = \lambda d + \bar{w}r, \quad \hat{Q} = \bar{d}\lambda + \hat{r}\hat{w} \]

Composite \( b, \hat{b} \) ghost – \((3L-3)\) insertions in \( L \)-loop amplitude

\[ b = -\frac{1}{4} \left( \frac{P^m\bar{\lambda}}{(\lambda\bar{\lambda})} + \frac{\bar{\lambda}\gamma_{mnpr}(d\gamma^{mnpr})}{384 (\lambda\bar{\lambda})^2} + \ldots \right), \]

so that

\[ [Q_{tot}, (b + \hat{b})] = H, \quad [Q_{tot}, (b - \hat{b})] = 0 \]

Integrate vertex insertions \( V(k_r, \tau_r) \) (functions of \((d, \theta)\))

For supergravity:

\[ V \sim P^m P^m G_{mn} + d^\alpha \hat{d}^\beta W_{\alpha\beta} + \ldots \]

superfields \( G_{mn}(X, \theta), W_{\alpha\beta}(X, \theta) \)
L-loop amplitude:

$$\int D\Phi D\hat{\Phi} \left( \mathcal{N}\hat{\mathcal{N}} \prod_{i=1}^{3(L-1)} \left( \int_0^{T_i} \frac{d\tau}{T_i} b \int_0^{T_i} \frac{d\tau}{T_i} \hat{b} \right) \int \prod_{r=1}^{4} d\tau r \ V(k_1, \tau_1) \ldots V(k_4, \tau_4) e^{-s} \right)$$

Regulator for large $\lambda, \lambda\bar{\lambda}$ divergences $\mathcal{N} = e^{-\lambda\bar{\lambda} + \theta r - \lambda ds + ...}$

$(3L-3)$ $b, \hat{b}$ insertions in L-loop amplitude

Saturation of fermionic zero modes. Requires detailed consideration of modes coming from regulator $\mathcal{N}\hat{\mathcal{N}}$, $b$ insertions and vertex insertions, $V$.

Constrains pattern of diagrams that contribute to amplitude.
Consider saturation of the 16 L fermionic $d$ zero modes.

11 L of these are soaked up by the $d'$ s in the factor $(s \, d)^{11L}$ leaving 5L $d$ zero modes to be soaked up by $b$ insertions and vertex operators.

**Period matrix**

$$IJ = \oint_{B_i} \omega J$$

1-form $$\omega_i = a_i^i \, d\tau_i$$ with $a_i^i = \pm 1$

B-cycles – basis of L loops

**Term in $b$ zero mode with most $d$ zero modes**

$$\int_0^{T_i} \frac{d\tau}{T_i} \, b \big|_{zero} = d^I_\gamma \, d^J_\delta \frac{\partial IJ}{\partial T_i}$$

$\gamma, \, \delta = 1, \ldots, 5$

Furthermore, $d^2_\alpha = 0$ so each $d$ component occurs only once
Three loops

\[ \partial^8 \mathcal{R}^4 \Lambda^{3D-20} \]

Ladder amplitude

The three-loop skeletons

\[ \partial^6 \mathcal{R}^4 \Lambda^{3(D-6)} \]

Two leading contributions

Not allowed
The five four-loop skeletons

to which vertices must be inserted

1
2
3
4
5
Four loops

Two of the five four-loop skeletons

Example of a leading amplitude

Ladder amplitude

$\partial^{12} \mathcal{R}^4 \Lambda^{4D-26}$

$\partial^8 \mathcal{R}^4 \Lambda^{4D-22}$
New feature arises at five loops. Require $3L-3 = 12$ b insertions. Can now get power

\[
\left( \frac{r}{\lambda \bar{\lambda}} \right)^{12}
\]

Small- $\lambda, \bar{\lambda}$ singularity and more than 11 $r'$ s, giving $0/0$. Needs new regulator (Berkovits-Nekrasov) changes systematics.
Five-loop skeletons
to which vertices must be inserted
Last two diagrams give leading contribution:

\[ \partial^8 \mathcal{R}^4 \Lambda^{5D-24} \]

Nonplanar  

Planar

Leading contributions obtained by attaching 4 vertices of the form \( P^m P^n G_{mn} \)

Gives four cancelled propagators, leaving the vacuum (skeleton) diagrams

\[ \int d^5D k (k^2)^{-12} \sim \Lambda^{5D-24} \]
Log UV divergence in $D=\frac{24}{5}$ dimensions proportional to

$$\partial^8 R^4 \log \Lambda$$

Furthermore, general arguments suggest higher loops also contribute to $\partial^8 R^4$ so this interaction is not protected.

c.f. If $\partial^8 R^4$ were protected the low energy five-loop amplitude would have behaved as

$$\partial^{10} R^4 \Lambda^{5D-26}$$

consistent with $D = 4 + \frac{6}{L}$ and UV finiteness of N=8 supergravity in D=4
Strongly suggests that supersymmetry protects interactions of the form $\partial^{2k} R^4$ up to $k=3$. The interaction $\partial^8 R^4$ is unprotected and likely to have an ultraviolet divergence at seven loops in $D=4$ dimensions.