Gravity as a Double Copy of Gauge Theory

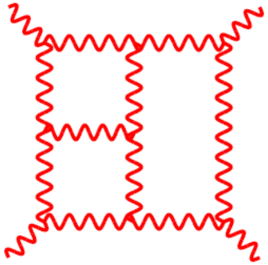
Cargese, June 25, 2010

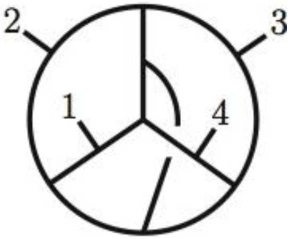
Zvi Bern, UCLA

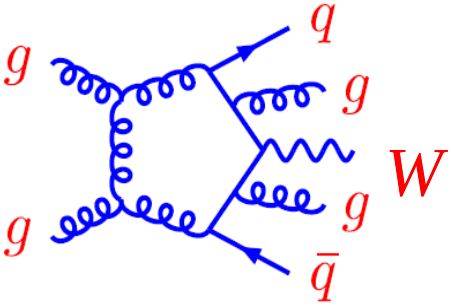
Lecture 2

Lecture 1: Scattering amplitudes in quantum field theories: On-shell methods, unitarity and twistors.

Lecture 2: Gravity as a double copy of gauge theory and applications to UV properties.







Outline

- **A new duality between color and kinematics.**
- **Gravity as a double copy of gauge theory.**
- **Review of conventional wisdom on UV divergences in quantum gravity – dimensionful coupling.**
- **Surprising one-loop cancellations: “no triangle property”.**
- **Additional observations and hints of UV finiteness.**
- **Explicit 3,4-loop calculations so we *know* the exact behavior.**
- **Origin of cancellation – high-energy behavior.**
- **Five loops and beyond.**

List of Papers

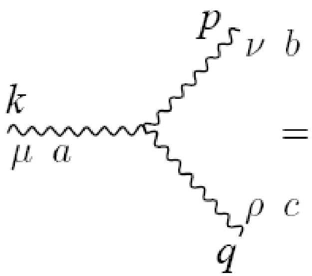
Research Articles:

- ZB, L. Dixon , R. Roiban, hep-th/0611086
- ZB, J.J. Carrasco, L. Dixon, H. Johansson, D. Kosower, R. Roiban, hep-th/0702112.
- ZB, J.J. Carrasco, D. Forde, H. Ita and H. Johansson, arXiv:0707.1035 [hep-th].
- ZB, J.J.M. Carrasco , L.J. Dixon, Henrik Johansson, R. Roiban, arXiv:0808.4112 [hep-th]
- ZB, J.J.M. Carrasco, H. Ita, H. Johansson, R. Roiban, arXiv:0903.5348 [hep-th]
- ZB, J.J.M. Carrasco, L.J. Dixon, H. Johansson, R. Roiban, arXiv:0905.2326 [hep-th]
- ZB, J. J. M. Carrasco, H.Johansson, arXiv:1004.0476 [hep-th]
- ZB, T. Dennen, Y.t. Huang and M. Kiermaier, arXiv:1004.0693 [hep-th].

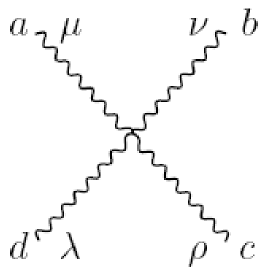
Review Articles:

- Z. Bern, gr-qc/0206071
- Z. Bern, J. J. M. Carrasco and H. Johansson, 0902.3765 [hep-th]
- H. Nicolai, Physics, 2, 70, (2009).
- R. P. Woodard, arXiv:0907.4238 [gr-qc].
- L. Dixon, arXiv:1005.2703 [hep-th].

Gauge Theory Feynman Rules



$$= -g f^{abc}\Big(\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu\Big)$$



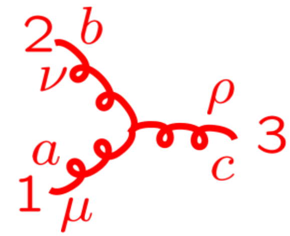
$$= \left\{ \begin{array}{l} -ig^2[f^{abe}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\ \quad + f^{ade}f^{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\ \quad + f^{ace}f^{ebd}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})] \end{array} \right.$$

Duality Between Color and Kinematics

ZB, Carrasco, Johansson

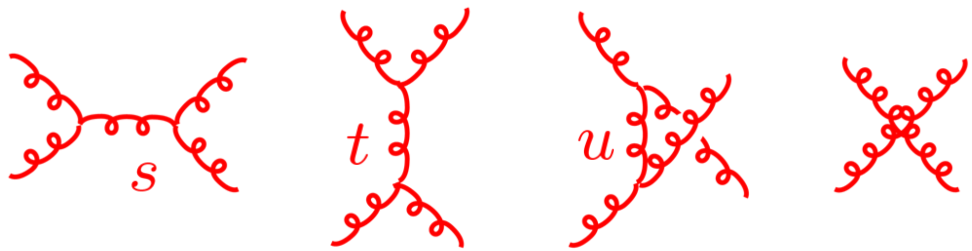
coupling constant → color factor → momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi identity $f^{a_1a_2b}f^{b,a_4,a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1 + k_2)^2$

$t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

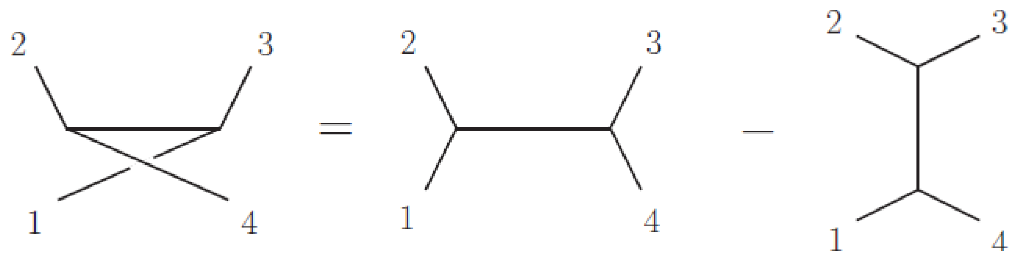
Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy similar identities

Four-Point Example

$$A_4(1^-, 2^-, 3^+, 4) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -i\frac{\langle 12\rangle^2[34]^2}{st} \equiv \frac{n_s}{s} + \frac{n_t}{t}$$

$$A_4(1^-, 4^+, 2^-, 3^+) = i\frac{\langle 12\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle} = -i\frac{\langle 12\rangle^2[34]^2}{tu} \equiv \frac{n_u}{u} - \frac{n_t}{t}$$



$$c_u = c_s - c_t$$
$$n_u = n_s - n_t$$

Choose e.g.: $$n_t = 0, \quad n_s = n_u = -i\frac{\langle 12\rangle^2[34]^2}{t}$$

Amplitude correct
Duality works!

- **At 4 points *any* choice which gives correct amplitudes works. Seems like a curiousity at 4 points.**
- **But at higher points it imposes rather non-trivial constraints on numerators. Does *not* work for Feynman diagrams. Must rearrange.**

Duality Between Color and Kinematics

BCJ

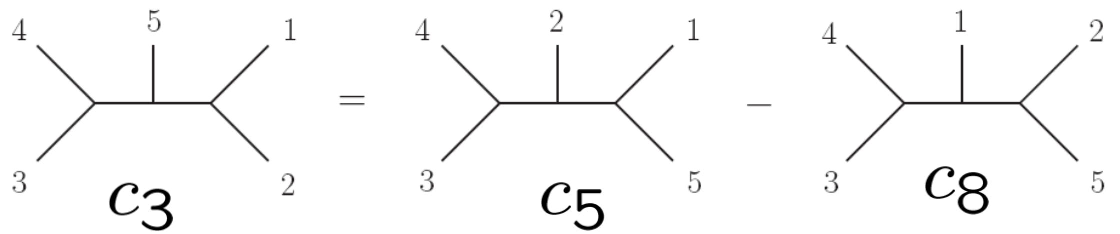
Consider five-point tree amplitude:

color factor

kinematic numerator factor

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{D_i}$$

Feynman propagators



$$c_3 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}\,, \qquad c_5 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}\,, \qquad c_8 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$\boxed{c_3 - c_5 + c_8 = 0 \quad \Leftrightarrow \quad n_3 - n_5 + n_8 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.

- **Color and kinematics satisfy same equations!**
- **Nontrivial constraints on amplitudes.**

There is now a partial string-theory understanding.

Bjerrum-Bohr, Damgaard, Vanhove; Stieberger; Mafra; Tye and Zhang

Duality Between Color and Kinematics

- **Conjecture states that numerators satisfying the duality can always be found.** ZB, Carraco, Johansson
- **Nontrivial consequences for amplitudes**

$$A_5^{\rm tree}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\rm tree}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\rm tree}(1,4,3,2,5)}{s_{13}s_{24}}$$

ZB, Carraco, Johansson

- **Proof of amplitude relations in both string theory and in field theory**
  - **— string theory based on monodromy.** Bjerrum-Bohr, Damgaard, Vanhove
  - **— field proof uses BCFW recursion.** Feng, Huang, Jia

Gravity vs Gauge Theory

Consider the gravity Lagrangian

flat metric

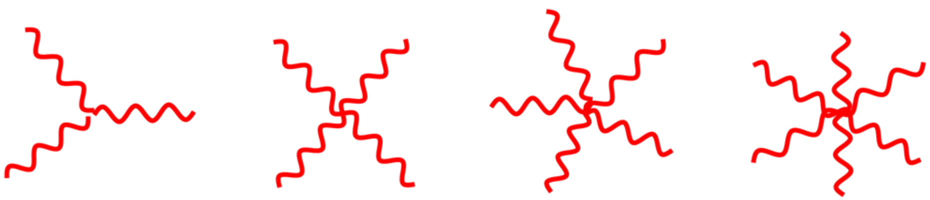
graviton field

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions



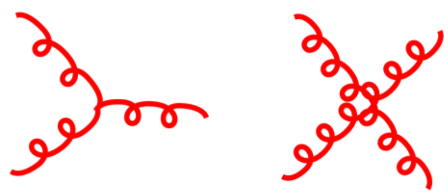
$+ \cdots$

terrible mess

Non-renormalizable

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

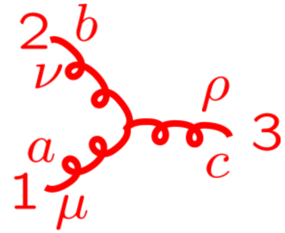


Only three- and four-point interactions

Gravity seems so much more complicated than gauge theory.

Three Vertices

Standard Feynman diagram approach.



Three-gluon vertex:

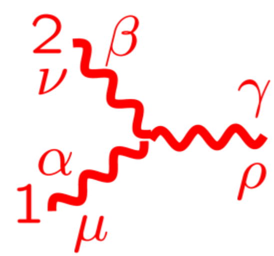
$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \eta_{\nu\rho}(k_1-k_2)_\mu + \eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\begin{aligned}&\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\ &+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\ &+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\ &+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]\end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

Definitely not a good approach.

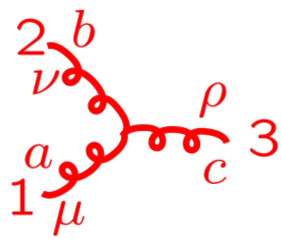
Simplicity of Gravity Amplitudes

People were looking at gravity the wrong way. On-shell viewpoint much more powerful.

On-shell **three vertices contains all information:**

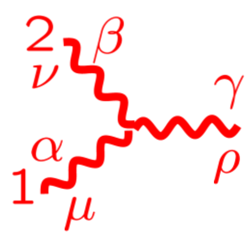
$$k_i^2 = 0$$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \\ \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

double copy of Yang-Mills vertex.

- **Using modern on-shell methods, any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.**
- **Higher-point vertices irrelevant! BCFW recursion for trees, BDDK unitarity method for loops.**

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

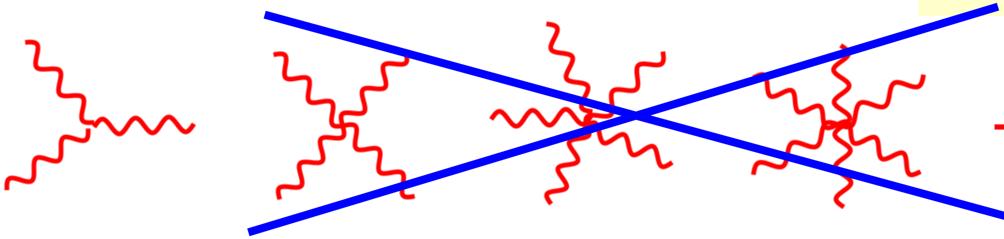
metric

flat metric

graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Infinite number of irrelevant interactions!

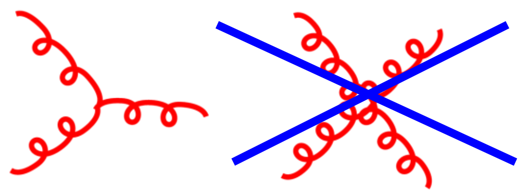


$+ \cdots$

Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



Only three-point interactions

Gravity seems so ~~much~~ no more complicated than gauge theory.

KLT Relations

A remarkable relation between gauge and gravity amplitudes exist at tree level which we will exploit.

At *tree level* Kawai, Lewellen and Tye have derived a relationship between closed and open string amplitudes.
In field theory limit, relationship is between gravity and gauge theor

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Gravity amplitude

Color stripped gauge theory amplitude

where we have stripped all coupling constants

$$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

Full gauge theory amplitude

Holds for any external states.
See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories

Recent field theory proof Bjerrum-Bohr, Damgaard, Feng, Sondergaard

Gravity and Gauge Theory Amplitudes

$$M_4^{\text{tree}}(1_h^-, 2_h^-, 3_h^+, 4_h^+) = \left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\text{tree}}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A_4^{\text{tree}}(1_g^-, 2_g^-, 4_g^+, 3_g^+)$$

gravity

gauge theory

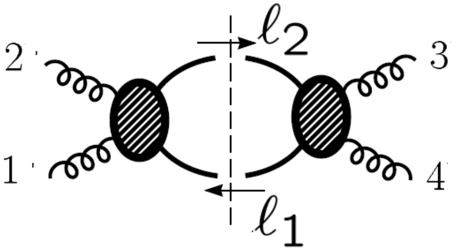
$$= \left(\frac{\kappa}{2}\right)^2 s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} \times \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,4\rangle\,\langle 4\,3\rangle\,\langle 3\,1\rangle}$$

$$\langle jl\rangle = \langle k_j^-|k_l^+\rangle = \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l) = \sqrt{2k_j\cdot k_l}\, e^{i\phi}$$

- **Holds for all states appearing in a string theory.**
- **Holds for all states of *N* = 8 supergravity.**

N = 8 Supergravity from N = 4 Super-Yang-Mills

Using unitarity and KLT we express cuts of N = 8 supergravity amplitudes in terms of N = 4 amplitudes.



$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times M_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1)$$

$$= s^2 \sum_{N=4 \text{ states}} \Big(A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(\ell_2, 3, 4, -\ell_1) \Big) \times \sum_{N=4 \text{ states}} \Big(A_4^{\text{tree}}(\ell_2, 1, 2, -\ell_1) \times A_4^{\text{tree}}(\ell_1, 3, 4, -\ell_2) \Big)$$

N = 8 susy sum factorizes

Key formula for N = 4 Yang-Mills two-particle cuts:

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st\, A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

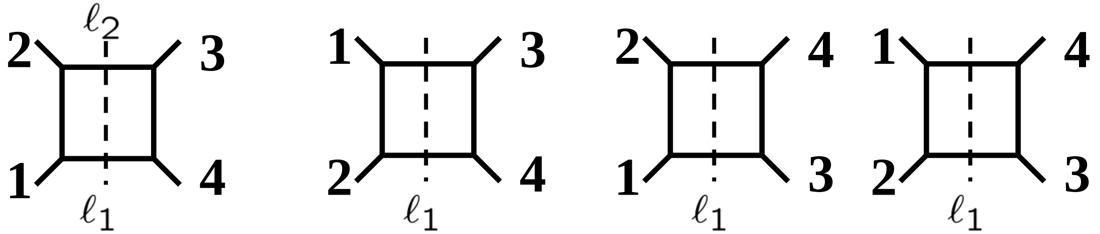
Key formula for N = 8 supergravity two-particle cuts:

Note recursive structure!

$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times M_4^{\text{tree}}(-\ell_2, \ell_3, -\ell_4, \ell_1)$$

$$= istu\, M_4^{\text{tree}}(1,2,3,4) \left[\frac{1}{(\ell_1 - k_1)^2} + \frac{1}{(\ell_1 - k_2)^2}\right] \left[\frac{1}{(\ell_2 - k_3)^2} + \frac{1}{(\ell_2 - k_4)^2}\right]$$

Generates all contributions with s-channel cuts.



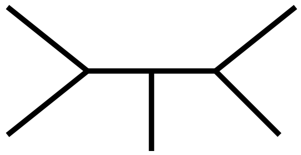
Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

gauge theory: $$\frac{1}{g^{n-2}}\mathcal{A}_n^{\rm tree}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

sum over diagrams with only 3 vertices

gravity: $$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\rm tree}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$



Holds if the n_i satisfy the duality. $\tilde{n}_i$ is from 2nd gauge theory

Gravity numerators are a double-copy of gauge theory ones!

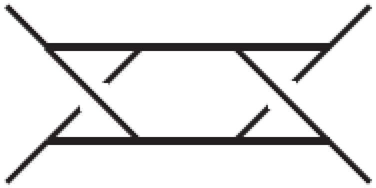
Proved using BCFW on-shell recursion relations that if duality holds, gravity numerators are 2 copies of gauge-theory ones.

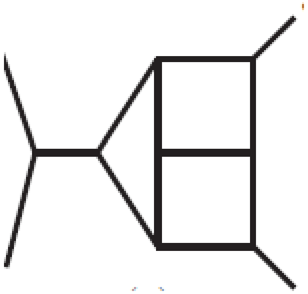
ZB, Dennen, Huang, Kiermaier

Cries out for a unified description of the sort given by string theory!

Loop-Level Generalization

ZB, Carrasco, Johansson (2010)





sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}_n^{\rm loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\rm loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

gravity

symmetry factor

Loop-level conjecture is identical to tree-level one except for symmetry factors and loop integration.
Double copy works if numerator satisfies duality.

The Double Copy in String Theory

Mafra; Tye and Zhang; Bjerrum-Bohr, Damgaard, Feng, Sondergaard,

open string

$$A_n \sim \int \frac{dx_1 \cdots dx_n}{\mathcal{V}_{abc}} \prod_{1\le i<j\le n} |x_i - x_j|^{k_i\cdot k_j} \exp\Big[\sum_{i<j}\Big(\frac{\epsilon_i\cdot\epsilon_j}{(x_i-x_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(x_i-x_j)}\Big)\Big]\Big|_{\text{multi-linear}}$$

closed string

$$M_n \sim \int \frac{d^2 z_1 \cdots d^2 z_n}{\Delta_{abc}} \prod_{1\le i<j\le n} (z_i - z_j)^{k_i\cdot k_j} \exp\Big[\sum_{i<j}\Big(\frac{\epsilon_i\cdot\epsilon_j}{(z_i-z_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(z_i-z_j)}\Big)\Big]$$
$$\times \prod_{1\le i<j\le n} (\bar z_i - \bar z_j)^{k_i\cdot k_j} \exp\Big[\sum_{i<j}\Big(\frac{\bar\epsilon_i\cdot\bar\epsilon_j}{(\bar z_i-\bar z_j)^2} + \frac{k_i\cdot\bar\epsilon_j - k_j\cdot\bar\epsilon_i}{(\bar z_i-\bar z_j)}\Big)\Big]\Big|_{\text{multi-linear}}$$

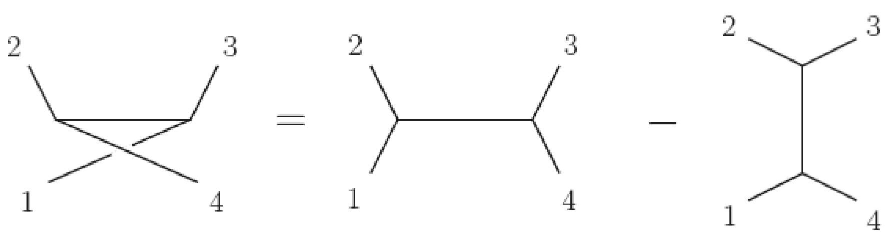
- **The closed-string Koba-Nielsen integrand is a double copy of the open string one. Well known fact.**
- **The double copy we are talking about is *after* carrying out the Koba-Nielsen integration.**
- **The heterotic string is a particularly good way to study the color-kinematics duality because of the parallel treatment of color and kinematics. But where does duality come from?**

Tye and Zhang

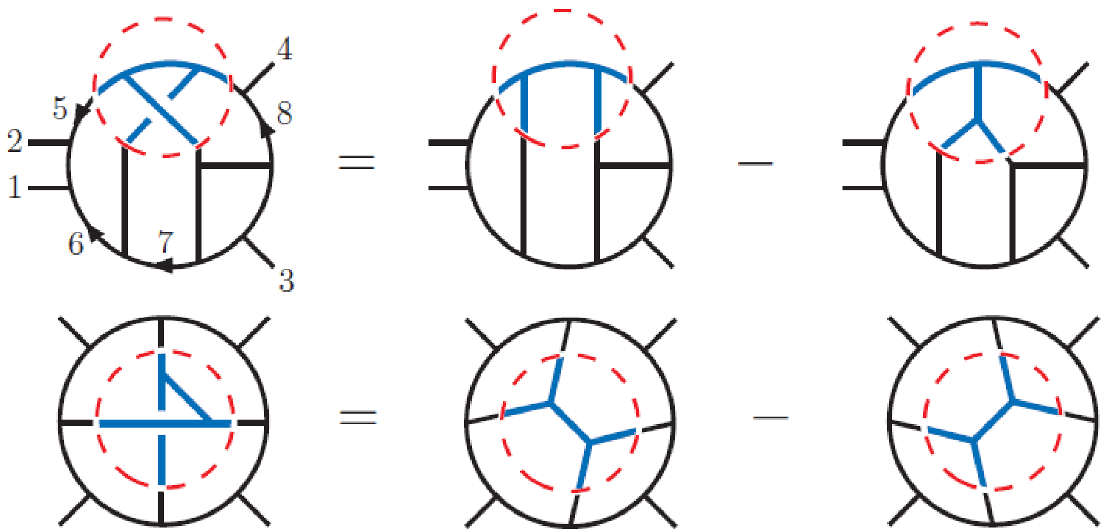
Relations Between Planar and Nonplanar

ZB, Carrasco, Johansson

Generally, planar is simpler than non-planar. Can we obtain non-planar from planar? The answer is yes!



Numerators satisfy identities similar to color Jacobi identities.



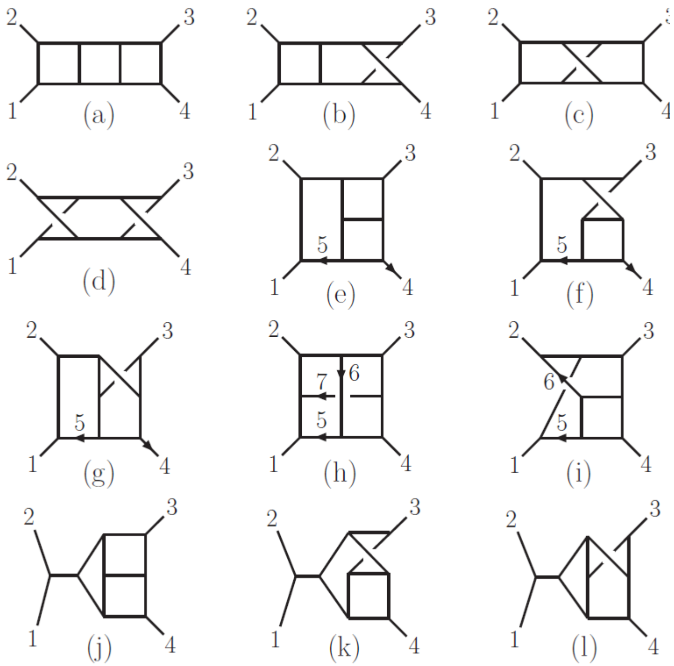
Numerator relations

Non-planar contributions can be derived from planar contributions.

Interlocking set of equations restrict numerators

Explicit Three-Loop Check

ZB, Carrasco, Johansson (2010)



$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k$$

For *N*=4 sYM we have the ability to go to high loop orders. Go to 3 loops. (1 & 2 loops works.)

Calculation very similar to earlier one with Dixon and Roiban, except now make the duality manifestly

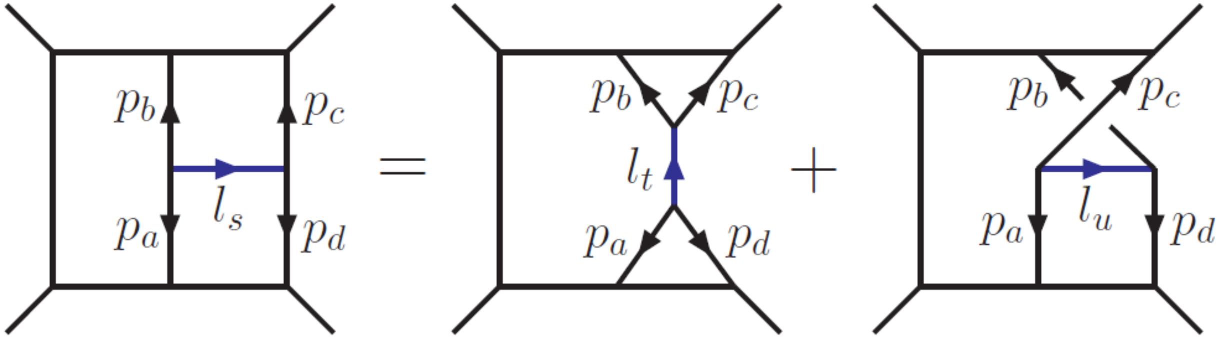
Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2 \big)/3$
(h)	$\big(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2 \big)/3$
(i)	$\big(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2 \big)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2k_i \cdot l_j$$

- **Duality works!**
- **Double copy works!**

Explicit Three-Loop Check

ZB, Carrasco, Johansson (2010)



$$
\begin{aligned}
n(\{\,V(p_a,p_b,l_s),\ V(-l_s,p_c,p_d),\ \cdots\}) = \\
n(\{\,V(p_d,p_a,l_t),\ V(-l_t,p_b,p_c),\ \cdots\}) \\
+n(\{\,V(p_a,p_c,l_u),\ V(-l_u,p_b,p_d),\ \cdots\})
\end{aligned}
$$

- **Easy to check the above relation holds.**
- **Must check all such relations.**

Lagrangians

ZB, Dennen, Huang, Kiermaier

$$L_{\rm YM} = \frac{1}{g^2} F^2 \qquad L_{\rm gravity} = \frac{2}{\kappa^2} \sqrt{-g}\, R$$

How can one take two copies of the gauge theory Lagrangian to give a gravity Lagrangian?

Add zero to the YM Lagrangian in a special way:

$$\mathcal{L}'_5 = -\tfrac{1}{2} g^3 \big(f^{a_1 a_2 b} f^{b a_3 c} + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c} \big) f^{c a_4 a_5} \times \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3 \mu} \frac{1}{\Box} (A^{a_4 \nu} A^{a_5 \rho}) \quad = 0$$

Through five points:

- **Feynman diagrams satisfy the color-kinematic duality.**
- **Introduce auxiliary field to convert contact interactions into three-point interactions.**
- **Take two copies: you get gravity!** $\quad A^\mu \tilde{A}^\nu \;\to\; h^{\mu\nu}$

At each order need to add more and more vanishing terms.

Lagrangians

One can continue this process but things get more complicated: Lagrangian six-point correction has ~100 terms.

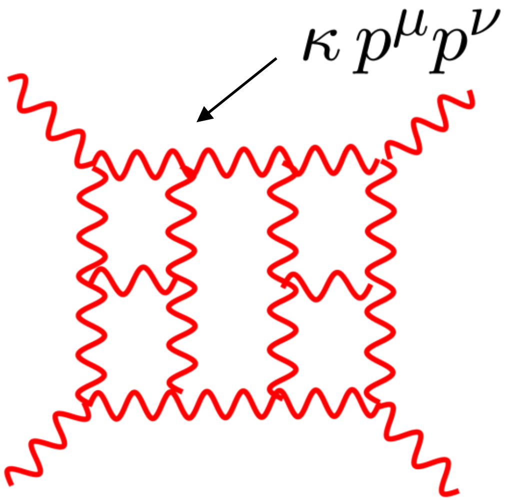
- **Is there a symmetry that restricts the terms?**
- **Can we understand the group theory structure of the vertices?**
- **Non-perturbative implications?**
- **Double copies of general classical solutions in gravity.**

$$g_{\mu\nu}(x) \sim \int dy\, A_\mu(x-y)\tilde{A}_\nu(y)$$

UV Properties of gravity

Power Counting at High Loop Orders

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Non-renormalizable by power counting.

Quantum Gravity at High Loop Orders

A key unsolved question is whether a finite point-like quantum gravity theory is possible.

- Gravity is non-renormalizable by power counting.

$$\kappa = \sqrt{32\pi G_N} \longleftarrow \text{Dimensionful coupling}$$

- Every loop gains $G_N \sim 1/M_{\mathrm{Pl}}^2$ mass dimension –2.

At each loop order potential counterterm gains extra

$$R^{\mu}{}_{\nu\sigma\rho} \sim g^{\mu\kappa}\partial_\rho\partial_\nu g_{\kappa\sigma} \quad \text{or} \quad D^2$$

- As loop order increases potential counterterms must have either more *R*'s or more derivatives

Divergences in Gravity

One loop: **Vanish on shell**

$R^2, \; R^2_{\mu\nu}, \; R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ **vanishes by Gauss-Bonnet theorem**

Pure gravity 1-loop finite (but not with matter) ‘t Hooft, Veltman (1974)

Two loop: Pure gravity counterterm has non-zero coefficient:

$$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\lambda\rho}$$

Any supergravity: Goroff, Sagnotti (1986); van de Ven (1992)

R^3 is ***not*** a valid supersymmetric counterterm.

Produces a helicity amplitude $(-,+,+,+)$ forbidden by susy.

Grisaru (1977); Tomboulis (1977)

The first divergence in *any* supergravity theory can be no earlier than three loops.

R^4 **Bel-Robinson tensor expected counterterm**

Deser, Kay, Stelle (1977); Kaku, Townsend, van Nieuwenhuizen (1977), Ferrara, Zumino (1978)

N = 8 Supergravity

The most supersymmetry allowed for maximum particle spin of 2 is $N = 8$. Eight times the susy of $N = 1$ theory of Ferrara, Freedman and van Nieuwenhuizen

We consider the N = 8 theory of Cremmer and Julia.

256 massless states

$N = 8$:	1	8	28	56	70	56	28	8	1
helicity :	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
	h^-	ψ_i^-	v_{ij}^-	χ_{ijk}^-	s_{ijkl}	χ_{ijk}^+	v_{ij}^+	ψ_i^+	h^+

Reasons to focus on this theory:

- With more susy suspect better UV properties.
- High symmetry implies technical simplicity.

Finiteness of *N* = 8 Supergravity?

We are interested in UV finiteness of *N* = 8 supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

- **Note: Perturbative finiteness is not the only issue for consistent gravity: nonperturbative completions. High energy behavior of theory.**
- **What symmetry or mechanism is powerful enough to render the theory finite?**

Opinions from the 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that *all* supergravity theories diverge (at three loops) has been widely accepted for over 25 years

R^4 **is expected counterterm**

Where is First Potential UV Divergence in D= 4 N = 8 Sugra?

Various opinions over the years:

3 loops	Conventional superspace power counting	Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985)
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N}$ = 6 harmonic superspace exists; algebraic renormalisation argument	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N}$ = 7 harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* $\mathcal{N}$ = 8 harmonic superspace exists; lightcone gauge locality arguments; Algebraic renormalization arguments	Grisaru and Siegel (1982); Howe, Stelle and Bossard (2009) Vanhove; Bjornsson, Green (2010) Kiermaier, Elvang, Freedman(2010) Ramond Kallosh (2010)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy	Kallosh; Howe and Lindström (1981)
9 loops	*Assume* Berkovits' superstring non-renormalization theorems can be carried over to D=4 $\mathcal{N}$ = 8 supergravity and extrapolate to 9 loops. Speculations on D=11 gauge invariance.	Green, Russo, Vanhove (2006) (retracted)

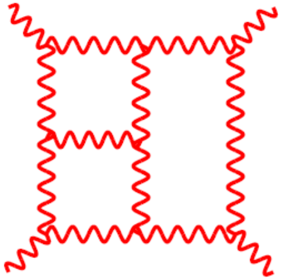
No divergence demonstrated above. Arguments based on lack of susy protection! We will present contrary evidence of all-loop finiteness.

To end debate, we need solid results!

Feynman Diagrams for Gravity

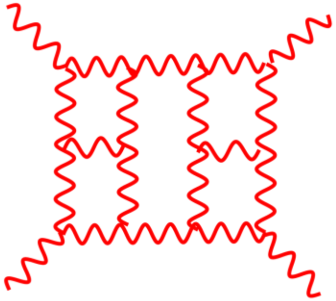
Suppose we want to put an end to the speculations by explicitly calculating to see what is true and what is false:

Suppose we wanted to check superspace claims with Feynman diagrams:



If we attack this directly get $\sim 10^{20}$ terms in diagram. There is a reason why this hasn't been evaluated.

In 1998 we suggested that five loops is where the divergence is:



This single diagram has $\sim 10^{30}$ terms prior to evaluating any integrals. More terms than atoms in your brain!

Using double copy property and unitarity method we can bypasses this Feynman diagram difficulties.

Novel $N = 8$ Supergravity UV Cancellations

A case that correct UV finiteness condition is:

$$\boxed{D < \frac{6}{L} + 4} \qquad (L > 1)$$

UV finite in $D = 4$
Same as $N = 4$ sYM!

D : dimension
L : loop order

Three pillars to our case:

- **Demonstration of *all*-loop order UV cancellations from "no-triangle property".** ZB, Dixon, Roiban
- **Identification of tree-level cancellations responsible for improved UV behavior.** ZB, Carrasco, Ita, Johansson, Forde
- **Explicit 3,4 loop calculations.** ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Key claim: The most important cancellations are generic to gravity theories. Supersymmetry helps make the theory finite, but is *not* the key ingredient for finiteness.

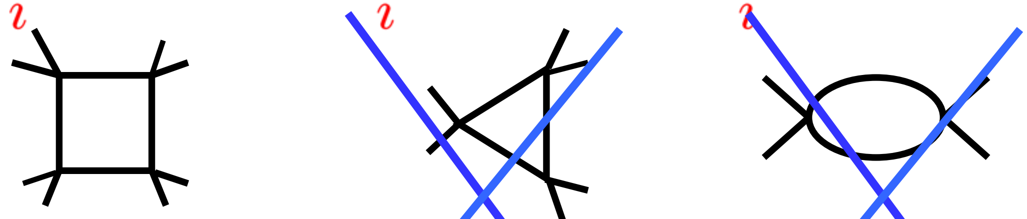
N = 8 Supergravity No-Triangle Property

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Proofs by Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

One-loop D = 4 theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

Brown, Feynman; Passarino and Veltman, etc

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$



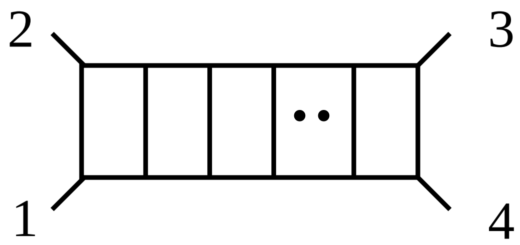
$$\int \frac{d^4 p}{(p^2)^4} \qquad \cancel{\int \frac{d^4 p}{(p^2)^3}} \qquad \cancel{\int \frac{d^4 p}{(p^2)^2}}$$

- **In N = 4 Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.**
- **The "no-triangle property" is the statement that same holds in N = 8 supergravity. Non-trivial constraint on analytic form of amplitudes.**
- **Unordered nature of gravity is important for this property**

Bjerrum-Bohr and Vanhove

$N = 8$ L-Loop UV Cancellations

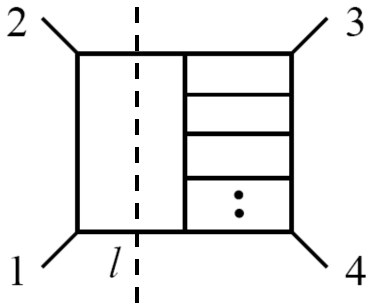
ZB, Dixon, Roiban



$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor

From 2 particle cut:

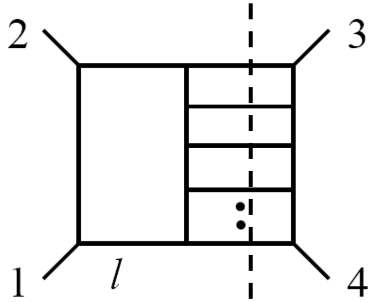


1 in $N = 4$ YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

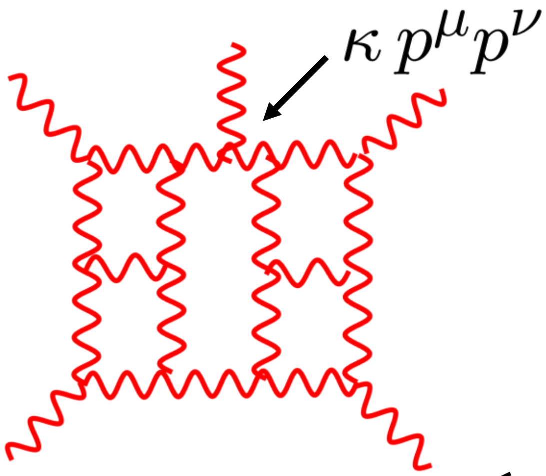
L-particle cut



- **Numerator violates one-loop "no-triangle" property.**
- **Too many powers of loop momentum in one-loop subamplitude.**
- **After cancellations behavior is same as in $N = 4$ Yang-Mills!**

- **UV cancellation exist to *all* loop orders! (not a proof of finiteness)**
- **These all-loop cancellations *not* explained by supersymmetry alone.**
- **Existence of these cancellations drive our calculations!**

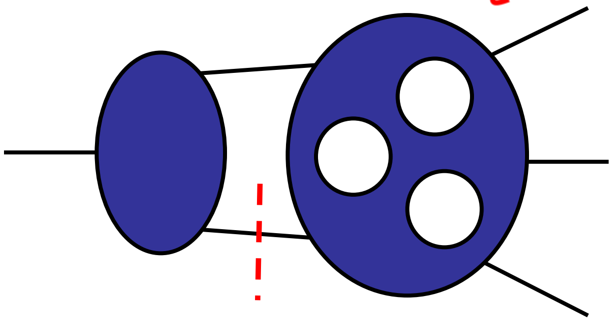
Higher-Point Divergences?



Add an extra leg:

1. **extra $\kappa\, p^\mu p^\nu$ in vertex**
2. **extra $1/p^2$ from propagator**

Adding legs generically does not worsen power count.



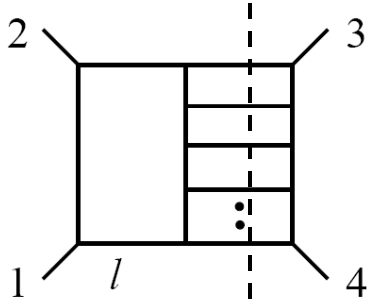
Cutting propagators exposes lower-loop higher-point amplitudes.

- **Higher-point divergences should be visible in high-loop four-point amplitudes.**
- **A proof of UV finiteness would need to systematically rule out higher-point divergences.**

Origin of Cancellations?

There does not appear to be a supersymmetry explanation for observed cancellations, especially as the loop order increases.

If it is *not* supersymmetry what might it be?



Tree Cancellations in Pure Gravity

Unitarity method implies all loop cancellations come from tree amplitudes. Can we find tree cancellations?

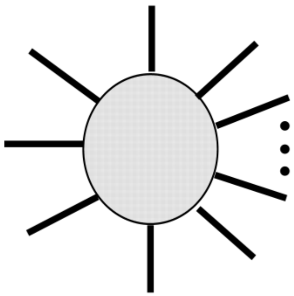
You don't need to look far: proof of BCFW tree-level on-shell recursion relations in gravity relies on the existence such tree cancellations! We know they exist.

Susy not required

Britto, Cachazo, Feng and Witten;
Bedford, Brandhuber, Spence and Travaglini
Cachazo and Svrcek; Benincasa, Boucher-Veronneau and Cachazo

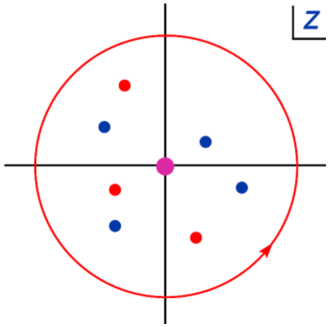
Consider the shifted gravity tree amplitude:

$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle, \qquad k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle,$$



How does $M(z)$ behave as $z \to \infty$?

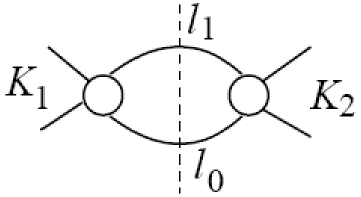
$$M(z) \to 0$$



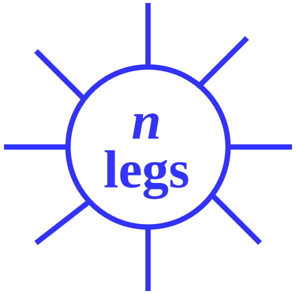
Loop Cancellations in Pure Gravity

ZB, Carrasco, Forde, Ita, Johansson

Powerful new one-loop integration method due to Forde makes it much easier to track the cancellations. Allows us to link one-loop cancellations to tree-level cancellations.



Observation: Most of the one-loop cancellations observed in *N* = 8 supergravity leading to "no-triangle property" are already present in non-supersymmetric gravity. Susy cancellations are on top of these.



$$(l^\mu)^{2n} \rightarrow (l^\mu)^{n+4} \times (l^\mu)^{-8}$$

Maximum powers of Loop momenta

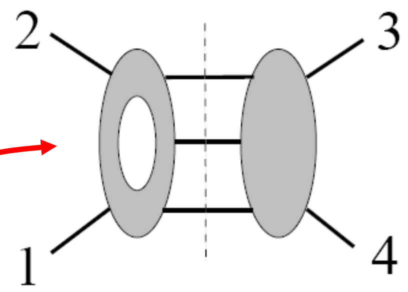
Cancellation generic to Einstein gravity

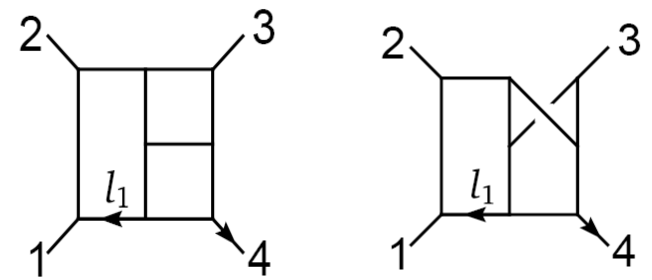
Cancellation from *N* = 8 susy

Proposal: This continues to higher loops, so that most of the observed *N* = 8 multi-loop cancellations are *not* due to susy but in fact are generic to gravity theories!

$\mathcal{N} = 8$ All Orders Cancellations

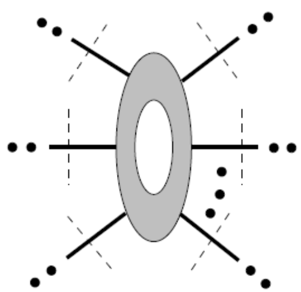
5-point
1-loop
known
explicitly





$$[(l_1 + k_4)^2]^2$$

must have cancellations between
planar and non-planar



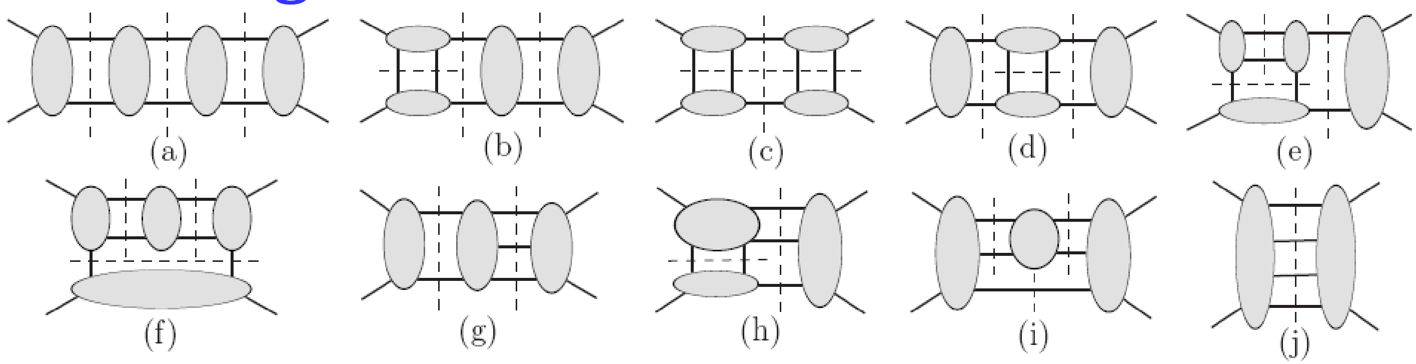
Using generalized unitarity and no-triangle hypothesis *any* one-loop subamplitude should have power counting of *N* = 4 Yang-Mills.

- Imposes non-trivial contraint on analytic structure of amplitudes.
- Cancellations powerful enough for UV finiteness
- Not a proof because all cuts need to be checked.

Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



For cut (g) have:

reduces everything to product of tree amplitudes

$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use Kawai-Lewellen-Tye tree relations

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12} A_4^{\text{tree}}(1,2,l_3,l_1) A_4^{\text{tree}}(2,1,l_3,l_1)$$

$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1 q_1} s_{l_3 q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\},$$

supergravity **super-Yang-Mills**

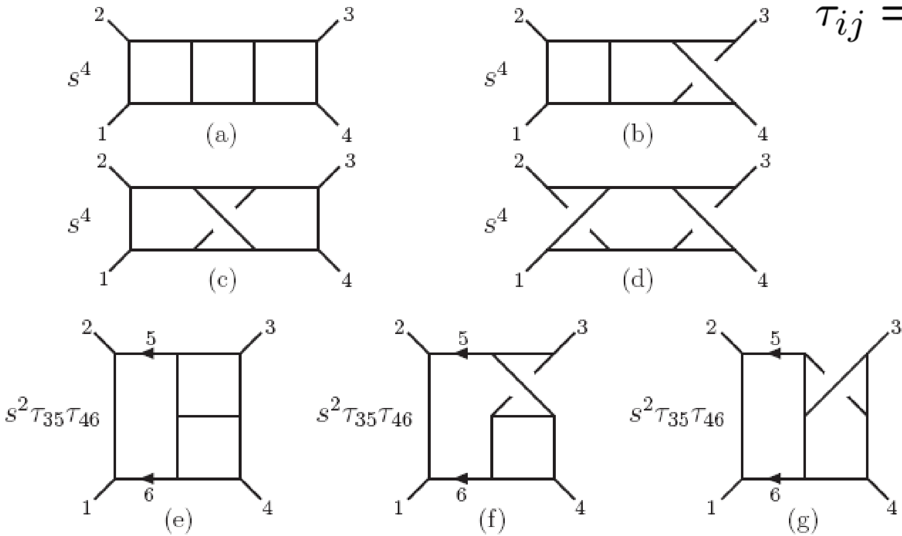
N = 8 supergravity cuts are sums of products of N = 4 super-Yang-Mills cuts

Complete Three-Loop $N = 8$ Supergravity Result

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\text{tree}} \sum_{S_3} \left[I^{(a)} + I^{(b)} + \frac{1}{2} I^{(c)} + \frac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \frac{1}{2} I^{(h)} + 2I^{(i)} \right]$$

$$\tau_{ij} = 2k_i \cdot k_j$$

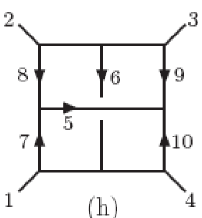


Three loops is not only UV finite it is "superfinite"— cancellations beyond those needed for finiteness in $D = 4$.

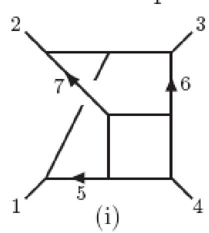
Finite for $D < 6$

All cancellations exposed.

$$\begin{aligned}&(s(\tau_{26}+\tau_{36}) + t(\tau_{15}+\tau_{25}) + st)^2 \\ &+ (s^2(\tau_{26}+\tau_{36}) - t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10}) \\ &+ s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) + t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10}) \\ &+ u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})\end{aligned}$$



$$\begin{aligned}&(s\tau_{45} - t\tau_{46})^2 - \tau_{27}(s^2\tau_{45} + t^2\tau_{46}) \\ &- \tau_{15}(s^2\tau_{47} + u^2\tau_{46}) - \tau_{36}(t^2\tau_{47} + u^2\tau_{45}) \\ &+ l_5^2 s^2 t + l_6^2 s t^2 - \tfrac{1}{3} l_7^2 stu\end{aligned}$$



$$\text{UVpole}_{D=6-2\epsilon} = \frac{1}{\epsilon} \frac{5\zeta_3}{(4\pi)^9} \left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\text{tree}}$$

Identical power count as $N = 4$ super-Yang-Mills

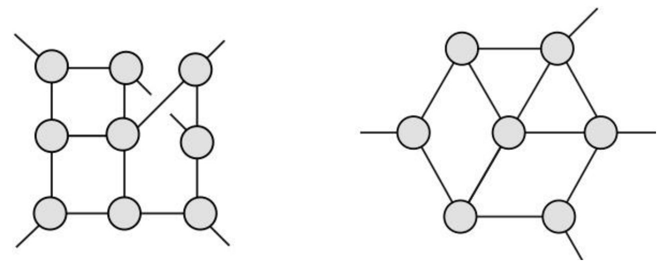
Four-Loop Construction

ZB, Carrasco, Dixon, Johansson, Roiban

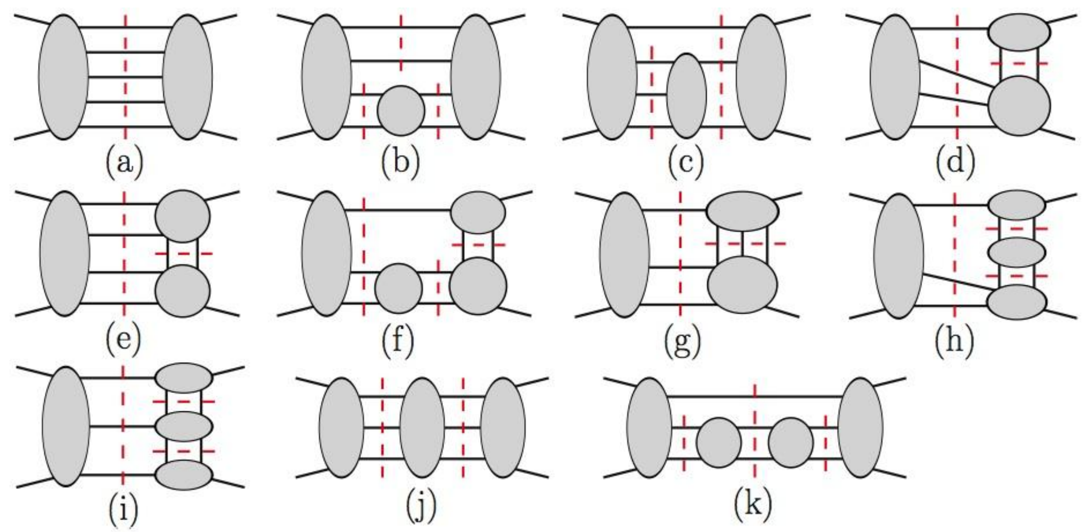
numerator

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \, \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

Determine numerators from 2906 maximal and near maximal cuts



Completeness of expression confirmed using 26 generalized cuts sufficient for obtaining the complete expression

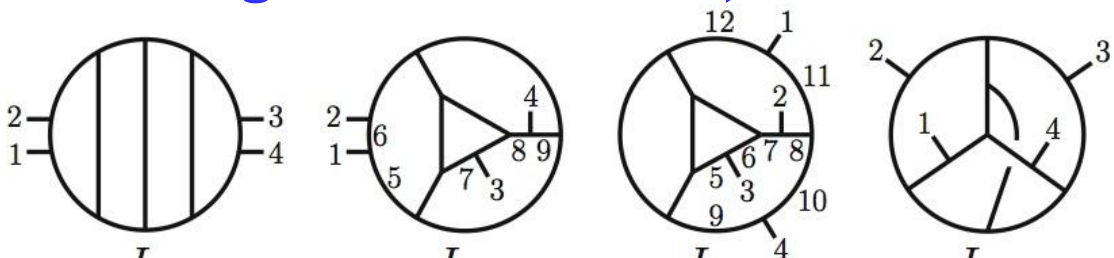


11 most complicated cuts shown

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



I_1 I_{25} I_{32} I_{50}

Journal submission has mathematica files with all 50 diagrams

$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

John Joseph shaved!
UV finite for $D < 5.5$
It's very finite!

"I'm not shaving until we finish the calculation"
— John Joseph Carrasco

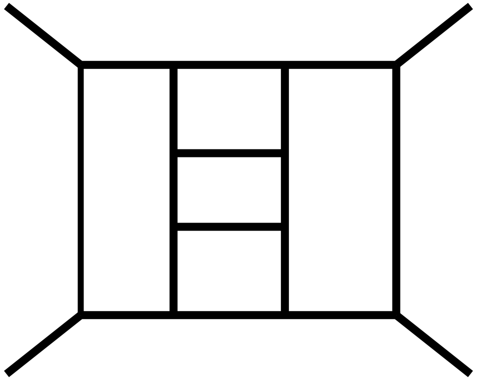
Recent world line formalism construction of Green and Bjornsson agrees with this count.

Five Loops is the New Challenge

• **Recent papers argue that susy protection cannot extend beyond 7 loops.** **See Michael Green's talk**

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Vanhove; Green and Bjornsson ; Kallosh and Ramond

• **If no other cancellations, this implies a worse behavior at 5 loops than for *N* = 4 sYM theory.**

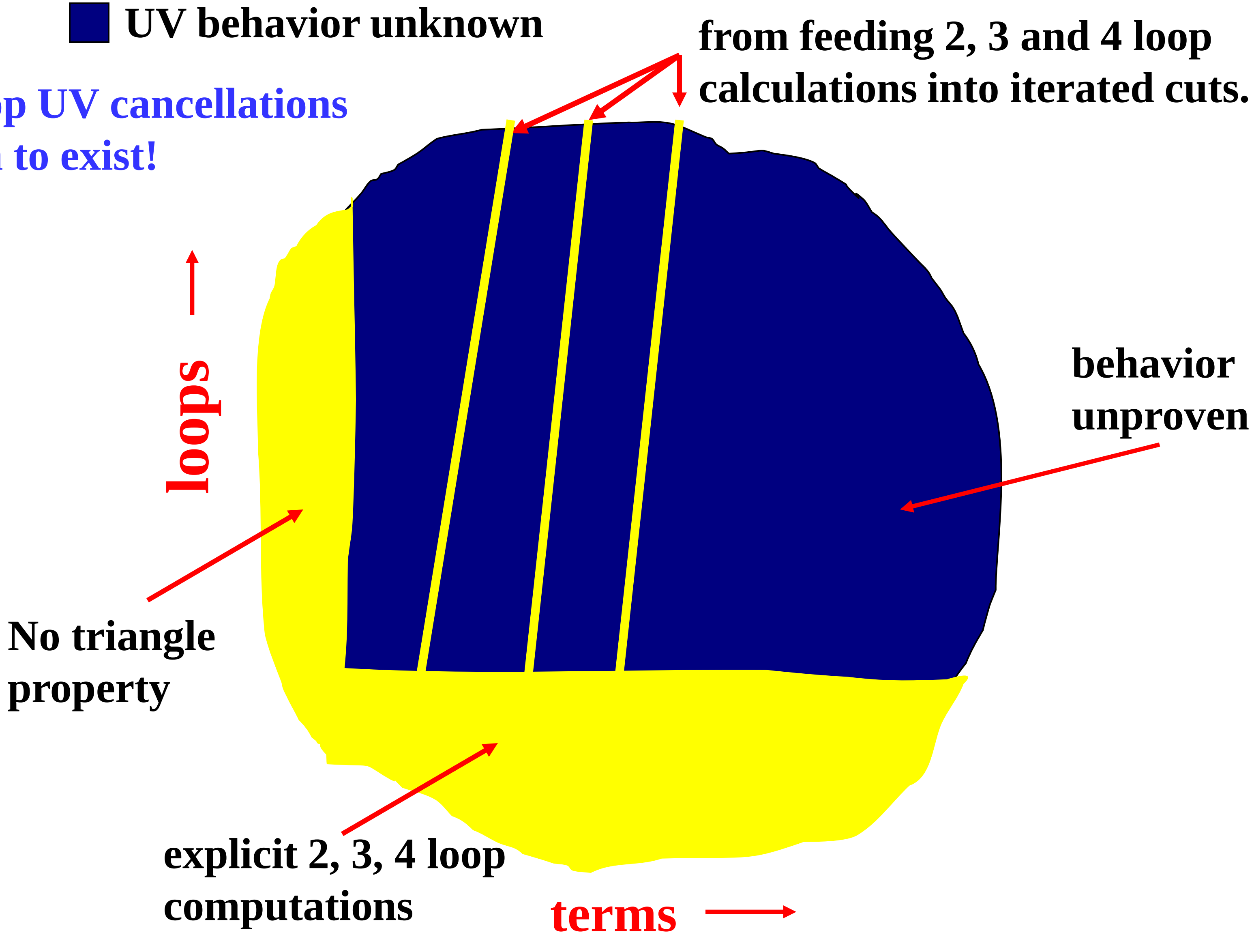


Schematic Illustration of Status

Same power count as $\mathcal{N}=4$ super-Yang-Mills

UV behavior unknown

All-loop UV cancellations known to exist!



Summary

- **New duality between color and kinematics**
- **Gravity as a double copy of gauge theory. Checked at 3 loops!**
- **Unitarity method gives us means of studying this at loop level. Extremely efficient way to calculate.**
- ***N* = 8 supergravity has UV cancellations with no known supersymmetry argument.**
  - **No-triangle property implies cancellations strong enough for finiteness to *all* loop orders, in a class of terms.**
- **At four points 2, 3, 4 loops, *established* that cancellations are complete and *N* = 8 supergravity has the same power counting as *N* = 4 Yang-Mills.**
- **Understanding 5 loops and beyond is the next challenge.**

Double copy property of gravity will surely continue to lead to an improved understanding of gravity theories.

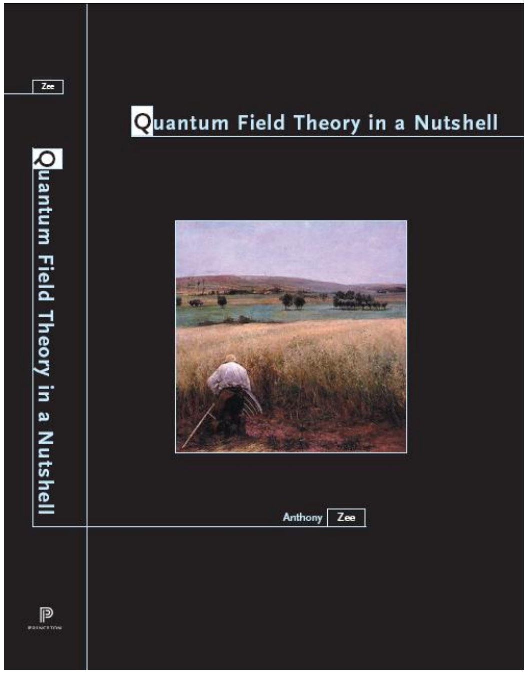
Reading List: Review Articles

- L. Dixon 1996 TASI lectures hep-ph/9601359 (trees basic ideas)
- Z. Bern, L. Dixon and D. Kosower, hep-ph/9602280 (early review on loops)
- Z. Bern, gr-qc/0206071 (early applications to gravity)
- F. Cachazo and P. Svrcek, hep-th/0504194 (twistors)
- Z. Bern, L. Dixon, D. Kosower, arXiv:0704.2798 (tools for QCD).
- Z. Bern, J.J.M. Carrasco, H. Johansson, arXiv:0902.3765 (UV properties of gravity)
- R. Woodard, arXiv:0907.4238 (gravity)
- L. Dixon , arXiv:1005.2703 (UV properties of gravity)

Further Reading

Hermann Nicolai, *Physics Viewpoint*, "Vanquishing Infinity"
http://physics.aps.org/articles/v2/70

Anthony Zee, *Quantum Field Theory in a Nutshell*,
2nd Edition is first textbook to contain modern
formulation of scattering and commentary
on new developments. 4 new chapters.



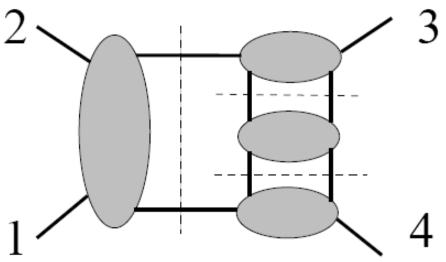
Some amusement

YouTube: Search “Big Bang DMV”, first hit.

Power Counting To All Loop Orders

Z.B., Dixon, Dunbar, Perelstein and Rozowsky

From '98 paper:



- **Assumed rung-rule contributions give the generic UV behavior.**
- **Assumed no cancellations with other uncalculated terms.**

- **No evidence was found that more than 12 powers of loop momenta come out of the integrals.**
- **This is precisely the number of loop momenta extracted from the integrals at two loops.**

Elementary power counting for 12 loop momenta coming out of the integral gives finiteness condition:

$$\boxed{D < \frac{10}{L} + 2} \qquad (L > 1)$$

In $D = 4$ finite for $L < 5$.
L is number of loops.

$D^4 R^4$ **counterterm expected in $D = 4$, for $L = 5$**

Finiteness Conditions

Through L = 3 loops the correct finiteness condition is ($L > 1$):

"superfinite" in D = 4

$$\boxed{D < \frac{6}{L} + 4}$$

same as N = 4 super-Yang-Mills

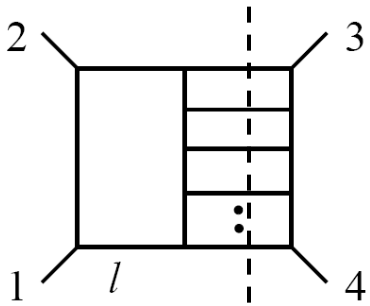
***not* the weaker result from iterated two-particle cuts:**

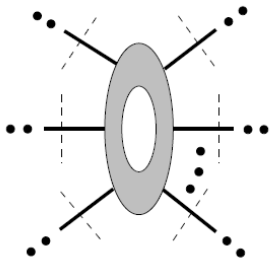
finite in D = 4 for L = 3,4

$$D < \frac{10}{L} + 2$$

(old prediction)

Beyond L = 3, as already explained, from special cuts we have strong evidence that the cancellations continue.





All one-loop subdiagrams should have same UV power-counting as N = 4 super-Yang-Mills theory.

No known susy argument explains these cancellations

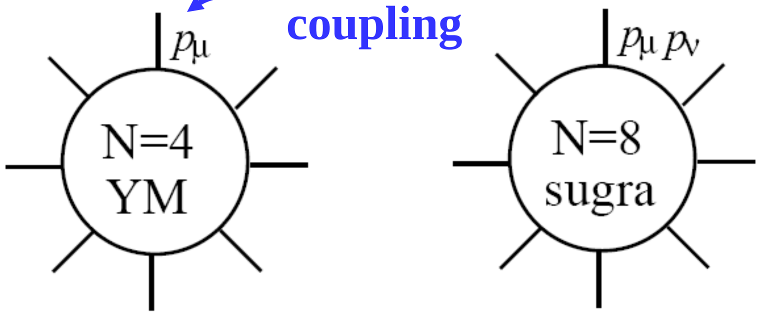
Cancellations at One Loop

Crucial hint of additional cancellation comes from one loop.

Surprising cancellations not explained by any known susy mechanism are found beyond four points

One derivative coupling

Two derivative coupling



ZB, Dixon, Perelstein, Rozowsky (1998);
ZB, Bjerrum-Bohr and Dunbar (2006);
Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager (2006)

Two derivative coupling means $N = 8$ should have a worse power counting relative to $N = 4$ super-Yang-Mills theory.

However, we have strong evidence that the UV behavior of both theories is the same at one loop.

Some Open Problems of Direct Interest

- Gravity as the square of YM. Better understanding needed!
- Better understanding of tree-level high energy behavior $1/z^2$. Key for improved UV behavoir at loop level.
- BCJ duality gravity relations. Who ordered this?
- Role of E7(7) symmetry? Kallosh, Arkani-Hamed, Cachazo, Kaplan

Comments on Consequences of Finiteness

- Suppose $N = 8$ SUGRA is finite to all loop orders. Would this prove that it is a **nonperturbatively** consistent theory of quantum gravity? **Of course not!**
- At least two reasons to think it needs a nonperturbative completion:

  — Likely $L!$ or worse growth of the order L coefficients,

  $$\sim \ L! \ (s/M_{\mathrm{Pl}}^2)^L$$

  — Different $E_{7(7)}$ behavior of the perturbative series (invariant!), compared with the $E_{7(7)}$ behavior of the mass spectrum of black holes (non-invariant!)
- Note QED is renormalizable, but its perturbation series has **zero radius of convergence in α: $\sim L!\ \alpha^L$**. But it has many point-like nonperturbative UV completions —asymptotically free GUTS.