Gravity as a Double Copy of Gauge Theory

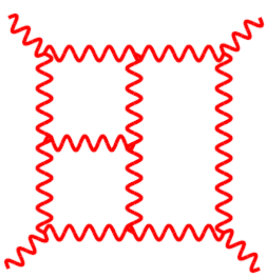
Cargese, 2010

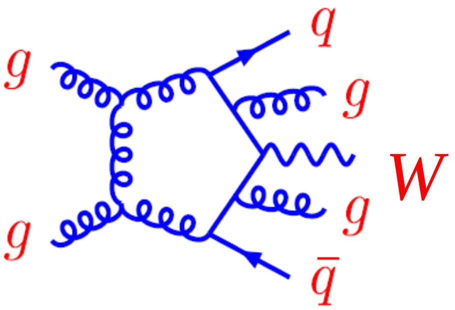
Zvi Bern, UCLA

Lecture 1

Lecture 1: Scattering amplitudes in quantum field theories: On-shell methods, unitarity and twistors.

Lecture 2: Gravity as a double copy of gauge theory and applications to UV properties.





Plan

"A method is more important than a discovery, since the right method can lead to new and even more important discoveries" -- L.D. Landau

Wish to discuss the general problem of scattering amplitudes in quantum field theory, before turning to the double-copy property of multi-loop gravity.

The basic issues and tools are the same in:

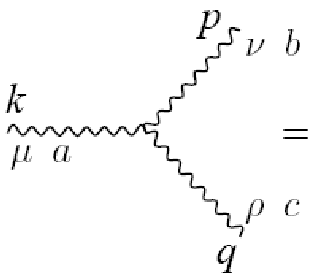
- **Precision phenomenology at the LHC.**
- **Weak coupling calculations of scattering amplitudes for the AdS/CFT correspondence.**
- **Unraveling the multi-loop structure of gauge and gravity amplitudes.**
- **UV properties of supergravity.**

Will discuss the modern 21st century tools and applications in this talk. **See Nima's talks for more theoretical aspects**

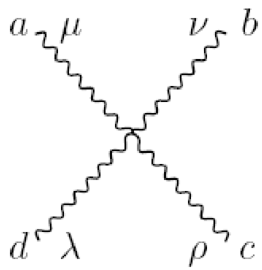
Topics

- Examples of applications of on-shell methods.
  - — State-of-the-art collider physics.
  - — AdS/CFT.
  - — UV properties of quantum gravity
- Spinors, twistors and amplitudes.
- MHV rules and on-shell recursion.
- Loop Amplitudes: Unitarity method.
- Duality between color and kinematics
- Double copy relation between gravity and gauge theory diagrams.
- Gravity.   UV properties of $N = 8$ gravity.

Gauge Theory Feynman Rules



$$= -g f^{abc}\Big(\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu\Big)$$



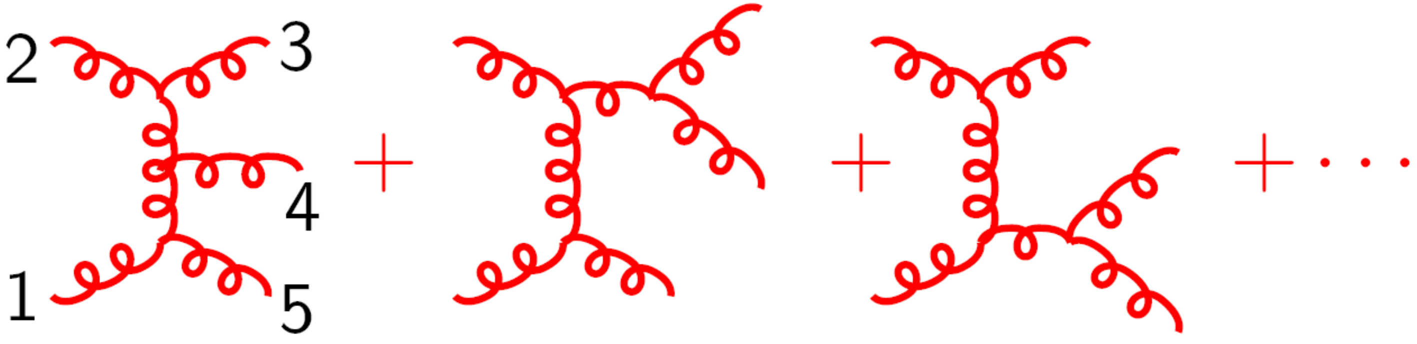
$$= \left\{ \begin{array}{l} -ig^2[f^{abe}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\ \qquad + f^{ade}f^{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\ \qquad + f^{ace}f^{ebd}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})] \end{array} \right.$$

Also fermions and ghosts

Color and kinematics mixed together

Tree-level example: Five gluons

Consider the five-gluon amplitude



If you evaluate this you find…

Result of evaluation (actually only a small part of it):

$$k_1 \cdot k_4\,\varepsilon_2 \cdot k_1\,\varepsilon_1 \cdot \varepsilon_3\,\varepsilon_4 \cdot \varepsilon_5$$

Spinors expose simplicity

Xu, Zhang and Chang
Berends, Kleis and Causmaeker
Gastmans and Wu
Gunion and Kunszt
& many others

Spinor helicity for massless polarization vectors:

particle momentum

$$\varepsilon^{+}_{\mu}(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\, k\rangle}, \qquad \varepsilon^{-}_{\mu}(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\, q]}$$

Reference momentum

Chinese magic

More sophisticated version of circular polarization: $\epsilon_\mu = (0, 1, \pm i, 0)$

All required properties of circular polarization satisfied:

$$\epsilon_i^2 = 0, \qquad k \cdot \epsilon_i = 0, \qquad \epsilon_i^+ \epsilon_i^- = -1$$

Changes in reference momentum q equivalent to on-shell gauge transformations:

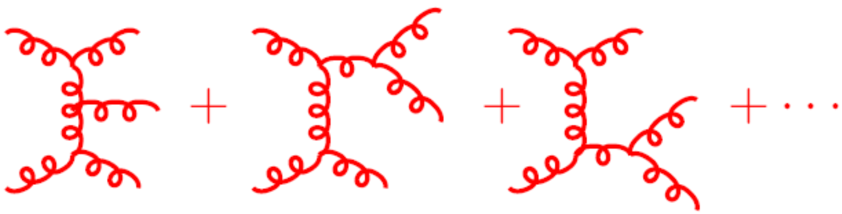
$$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\, l\rangle = \langle k_{j\,-} | k_{l\,+}\rangle = \sqrt{2k_j \cdot k_l}\; e^{i\phi} \;= \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l)$$

$$\epsilon_{\dot a \dot b}\tilde\lambda_j^{\dot a}\tilde\lambda_l^{\dot b} \longleftrightarrow [j\, l] = \langle k_{j\,+} | k_{l\,-}\rangle = -\sqrt{2k_j \cdot k_l}\; e^{-i\phi} \;= \tfrac{1}{2}\bar{u}(k_j)(1-\gamma_5)u(k_l)$$

Graviton polarization tensors are squares of these:

$$\epsilon^{+}_{\mu\nu} = \epsilon^{+}_{\mu}\epsilon^{+}_{\nu} \qquad\qquad 2 = 1 + 1$$

Reconsider Five Gluon Tree



With a little Chinese magic:

$$A_5^{\text{tree}}(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$A_5^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^+) = i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

These are color stripped amplitudes:

$$\mathcal{A}_5 = \sum_{\text{perms}} \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})\, A_5(1,2,3,4,5)$$

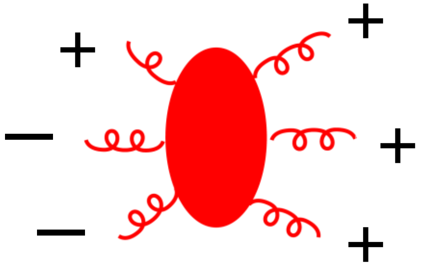
Motivated by the Chan-Paton color organization of open string amplitudes.

Mangano and Parke

Parke and Taylor (1984)

MHV Amplitudes

At tree level Parke and Taylor conjectured a very simple form for *n*-gluon scattering.

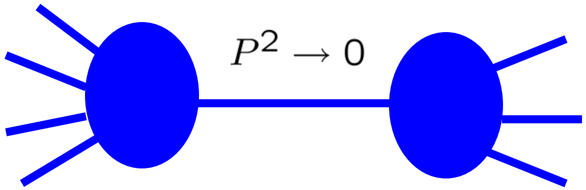


$$A(1^-,2^-,3^+,\ldots,n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

$$\mathcal{A}(1^-,2^-,3^+,\ldots,n^+) = \sum_{\text{perms}} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A(1^-,2^-,3^+,\ldots,n^+)$$

This was guessed by calculating low points and then finding a formula with correct kinematic poles in all channels.

Proven by Berends and Giele

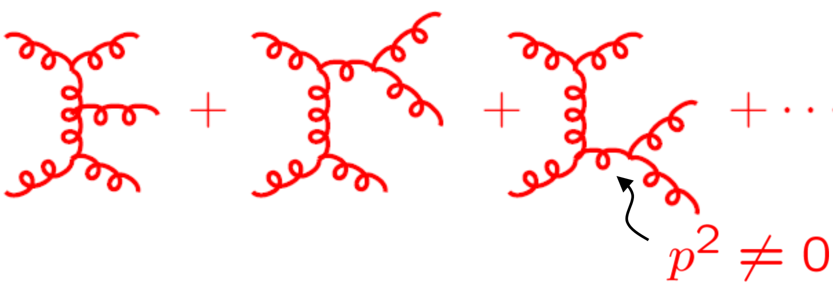


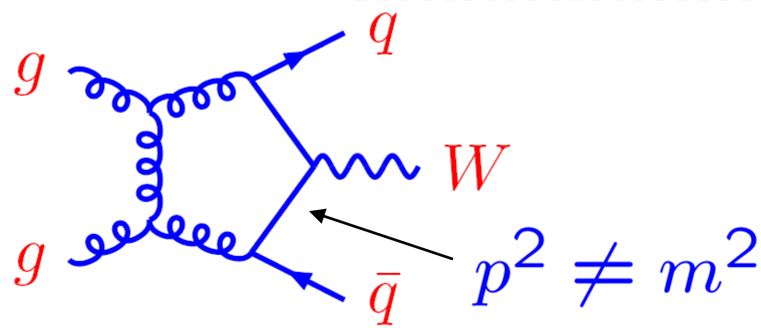
This simplicity has echoes for general helicities and at loop level.

ZB, Dixon, Dunbar, Kosower
Cachazo, Svrcek, Witten; ZB,Dixon, Kosower
Brandhuber, Spence and Travaglini

Why are Feynman diagrams clumsy for high loop or multiplicity processes?

- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.





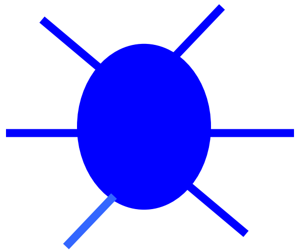
- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

- **All steps should be in terms of gauge invariant on-shell states. $p^2 = m^2$ On shell formalism.**
- **Radical rewrite of gauge theory needed.**

Off-shell Formalisms

In graduate school you learned that scattering amplitudes need to be calculated using unphysical gauge dependent quantities: off-shell Green functions

$$p^2 \neq m^2$$

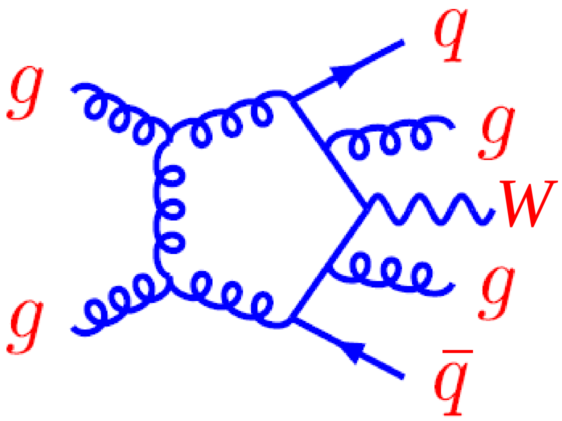


Standard machinery:

- Fadeev-Popov procedure for gauge fixing.
- Taylor-Slavnov Identities.
- BRST.
- Gauge fixed Feynman rules.
- Batalin-Fradkin-Vilkovisky quantization for gravity.
- Off-shell constrained superspaces.

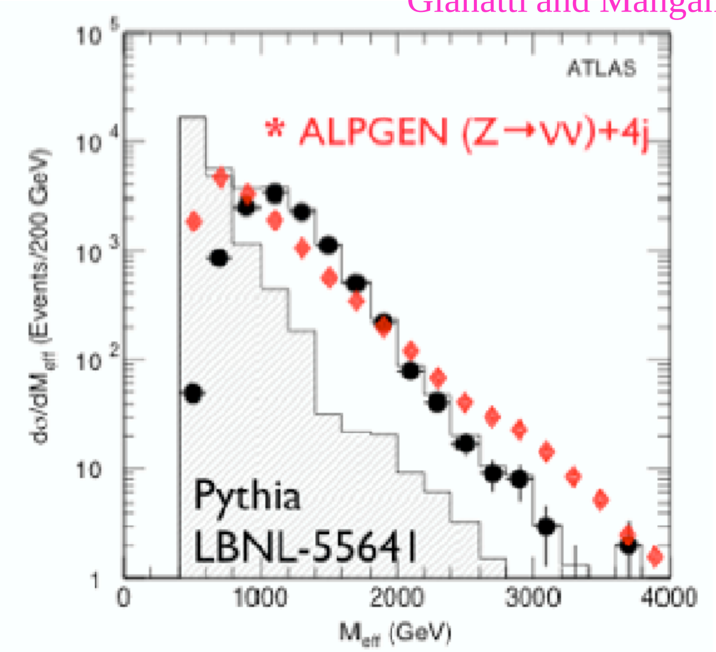
We won't need any of this. We will reformulate perturbative quantum field theory in terms of on-shell quantities.

State-of-the-art scattering amplitudes for LHC Physics



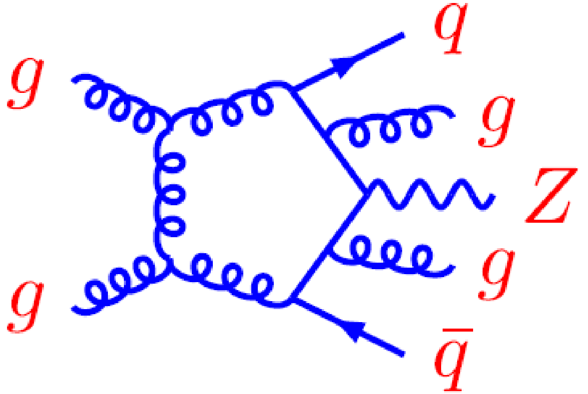
Example: Susy Search

Gianatti and Mangano



Early ATLAS TDR studies using PYTHIA overly optimistic.

- **ALPGEN is based on LO matrix elements and much better at modeling hard jets.**
- **What will disagreement between ALPGEN and data mean for this plot? Hard to tell. Need NLO.**

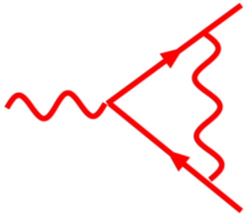


Such a calculation is *well beyond* anything that has been calculated using Feynman diagrams

We need $pp \to Z + 4$ jets at NLO

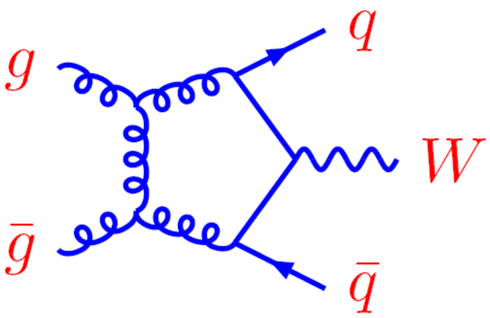
State-of-the-Art Feynman Diagram Calculations

In 1948 Schwinger computed anomalous magnetic moment of the electron.



In 2010 typical 1-loop modern Feynman diagram example:

$$gg \to W + 2 \text{ jets}$$



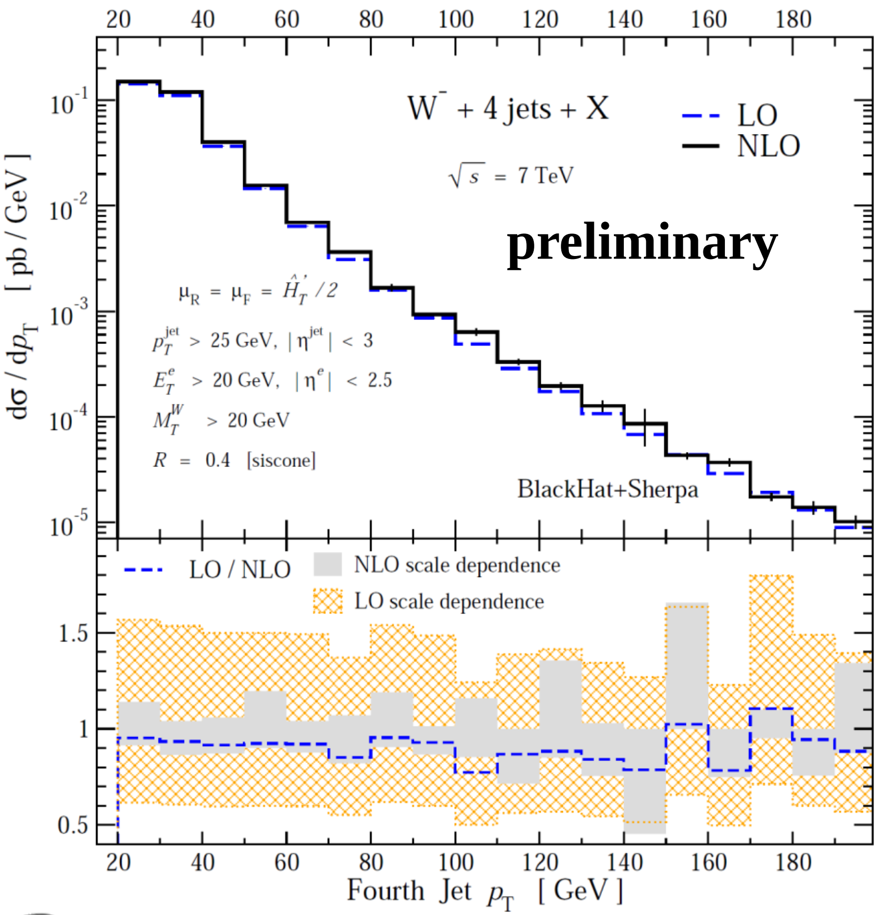
g **: gluon,** ***q*****: quark,** ***W, Z*** **: Weak boson**

60 years later at 1 loop only 2 (and sometimes 3) legs more than Schwinger!

Example: NLO QCD W + 4 Jets

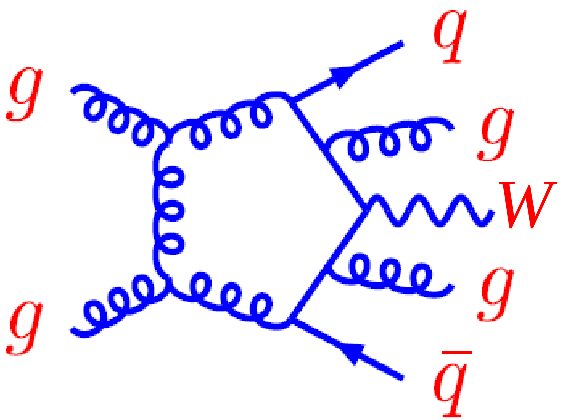
Berger, ZB, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre

Progress from the BlackHat Collaboration:



4th jet P_T

leading color virtual



BlackHat based on on-shell methods outlined in this lecture

- **Triumph of on-shell methods!**

Applications to AdS/CFT

$\mathcal{N}$ = 4 Super-Yang-Mills to All Loops

Since 't Hooft's paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. $\mathcal{N}$ = 4 sYM is much more promising.

- Special theory because of AdS/CFT correspondence.
- Maximally supersymmetric – 1 gluon, 4 gluinos, 6 scalars.
- Simplicity both at strong and weak coupling.

Remarkable relation

Alday and Maldacena

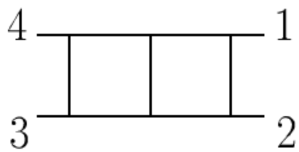
scattering at strong coupling in $\mathcal{N}$ = 4 sYM $\longleftrightarrow$ classical string theory in AdS space

To make this link need to evaluate $\mathcal{N}$ = 4 super-Yang-Mills amplitudes to *all* loop orders. Seems impossible even with modern methods.

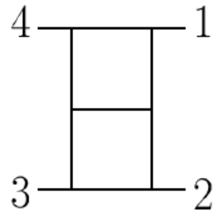
Loop Iteration of the $N = 4$ Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification. Computed via unitarity method.

$$-\,st\,A_4^{\text{tree}}\left\{\, s\right.$$



$$+\; t$$



$$\left.\right\}$$

ZB, Rozowsky, Yan

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\left(M_4^{\text{1-loop}}(s,t)\right)^2 + f(\epsilon)\, M_4^{\text{1-loop}}(s,t)\big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}} \qquad f(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$$

Anastasiou, ZB, Dixon, Kosower

$f(\epsilon)$ is universal function related to IR singularities $\qquad D = 4 - 2\epsilon$

This gives two-loop four-point planar amplitude as iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

ZB, Dixon, Smirnov

All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?

constant independent of kinematics.

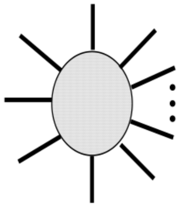
$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude



"BDS conjecture"

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

- **IR singularities agree with Magnea and Sterman formula.**
- **Limit of collinear momenta gives us key analytic information, at least for MHV amplitudes.**

Gives a definite prediction for *all* values of coupling given BES integral equation for the cusp anomalous dimension.

Beisert, Eden, Staudacher

Alday and Maldacena Strong Coupling

For MHV amplitudes: $F_4^{\text{1-loop}} = \frac{1}{2}\ln^2(s/t) + \frac{2\pi^2}{3}$ ZB, Dixon, Smirnov

constant independent of kinematics.

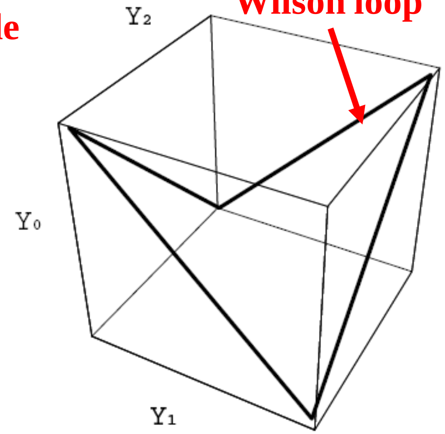
$$\mathcal{A}_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_4^{\text{1-loop}} + C\right]$$

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude



In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at *strong coupling* from an AdS string theory computation. Minimal surface calculation.

Very suggestive link to Wilson loops even at weak coupling.

Drummond, Korchemsky, Sokatchev ; Brandhuber, Heslop, and Travaglini
ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich

- **Identication of new symmetry: "dual conformal symmetry"**
- **Link to integrability**
- **Yangian structure!**

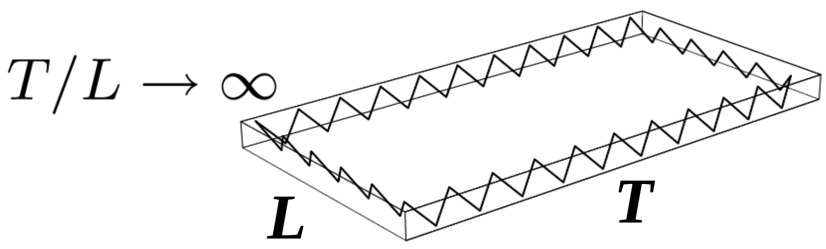
Drummond, Henn, Korchemsky, Sokatchev ;Berkovits and Maldacena; Beisert, Ricci, Tseytlin, Wolf Brandhuber, Heslop, Travaglini; Drummond, Henn, Plefka; Bargheer, Beisert, Galleas, Loebbert, McLoughlin.

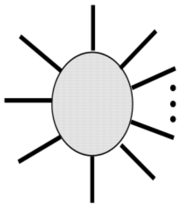
Trouble at Higher Points

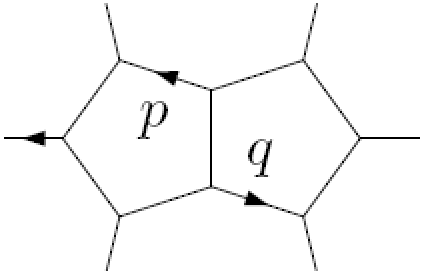
For various technical reasons it is non-trivial to solve for minimal surface for large numbers of gluons.

Alday and Maldacena realized certain terms can be calculated at strong coupling for an infinite number of gluons.

Disagrees with BDS conjecture







May also be trouble also in the Regge limit.

Bartels, Lipatov, Sabio Vera; Del Duca, Duhr and Glover; Brower, Nastase , Schnitzer, Tan

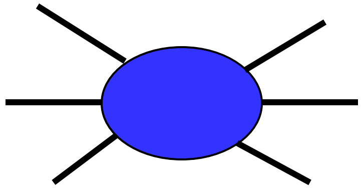
Explicit computation at 2-loop 6 points.

Need to modify conjecture!

ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich
Drummond, Henn, Korchemsky, Sokatchev

Can the BDS conjecture be repaired for six and higher points?

In Search of the Holy Grail



log of the amplitude

discrepancy

$$A^{\text{truth}} = A^{\text{div}} + A^{\text{BDS}} + R$$

Can we figure out the discrepancy?

Important new information from strong coupling

Explicit solution at eight points

$$k_i = x_{i+1} - x_i$$

$$A_{BDS} = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1, j\neq i, i-1}^{n} \log\frac{x_j^+ - x_i^+}{x_{j+1}^+ - x_i^+}\log\frac{x_j^- - x_{i-1}^-}{x_j^- - x_i^-}$$

Alday and Maldacena (2009)

$$A = A_{div} + A_{BDS} + R$$

$$R = -\frac{1}{2}\log(1+\chi^-)\log(1+\frac{1}{\chi^+}) + \frac{7\pi}{6} + \int_{-\infty}^{\infty} dt \frac{|m|\sinh t}{\tanh(2t+2i\phi)}\log\left(1+e^{-2\pi|m|\cosh t}\right)$$

Solution only valid for strong coupling and special kinematics

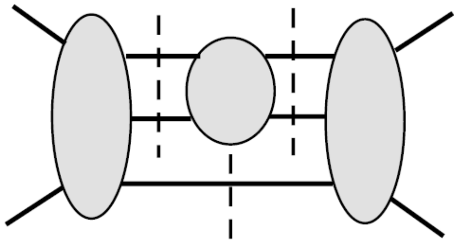
Last week paper from Alday, Gaiotto, Maldacena, Sever, Vieira has made exciting progress constraining the remainder function.

Third Application: Uncovering the structure of perturbative quantum gravity

Second lecture will focus on two issues:

- **Gravity as a double copy of gauge theory**
- **Remarkably good UV properties of $N = 8$ supergravity**

Basics of On-shell Methods



Twistors

In a remarkable paper Ed Witten demonstrated that twistor space reveals a hidden structure in scattering amplitudes.

Link is for N = 4 super-Yang-Mills theory, but at tree level hardly any difference from QCD.

Early work from Nair

Penrose twistor transform:

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2 \widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j i \mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}} \right) A(\lambda_i, \widetilde{\lambda}_i)$$

See Nima's talks

Witten's remarkable twistor-space link:

Witten; Roiban, Spradlin and Volovich

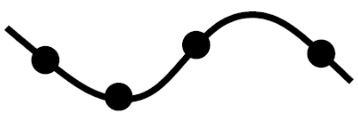
QCD scattering amplitudes $\longleftrightarrow$ **Topological String Theory**

Here we will discuss only the field theory consequences

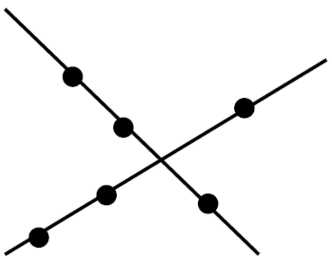
Amazing Simplicity

Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

$$d = q - 1 + L, \quad q = \#\ \text{negative helicities}, \quad L = \#\ \text{loops},$$



Connected picture



Disconnected picture

Structures imply an amazing simplicity in the scattering amplitudes. Massless gauge theory tree amplitudes are much much simpler than anyone imagined.

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena Bern and Kosower

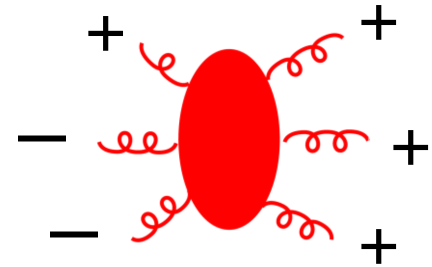
Remarkably gravity is similar, except derivative of delta function support instead of delta-function support.

MHV Rules

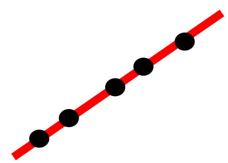
Cachazo, Svrcek and Witten

Consider MHV amplitudes.

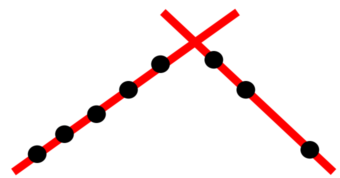
$$A(1^-,2^-,3^+,\ldots,n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

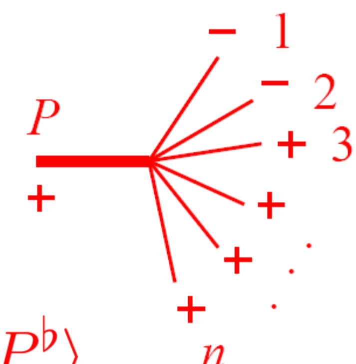


Supported on a straight line in twistor space

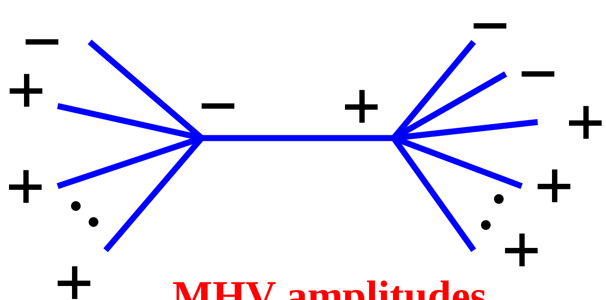


Non-MHV amplitudes supported on intersecting lines





In momentum space suggests MHV amplitudes are vertices for building new amplitudes.



MHV amplitudes at vertices

$\langle aP\rangle \to \langle a|\not{P}|q\rangle$ or $\langle aP\rangle \to \langle aP^\flat\rangle$

$$q^2 = 0 \qquad\qquad P^\flat \equiv P - \frac{P^2}{P\cdot q}\,q$$

Arbitrary null momentum

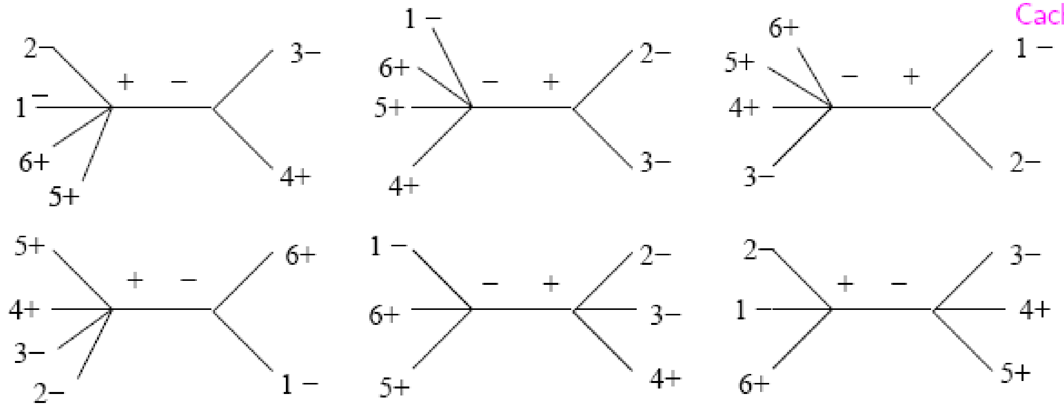
Massless

Six-gluon example

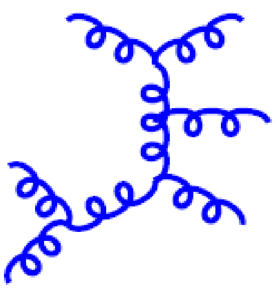
Cachazo, Svrcek and Witten

QCD gluon scattering amplitude

$- \; - \; - \; + \; + \; +$



$$
\begin{aligned}
A_6(1^-,2^-,3^-,4^+,5^+,6^+) = {} & \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle} \\
& + \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle} \\
& + \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle} \\
& + \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle} \\
& + \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle} \\
& + \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}
\end{aligned}
$$



220 Feynman diagrams

A "perfect" calculation

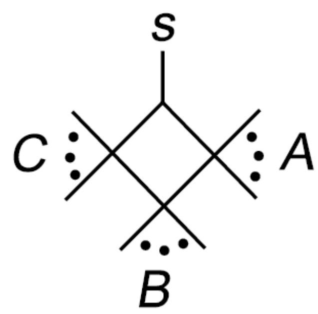
Twistor Structure at One Loop

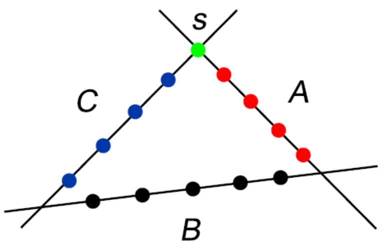
At one-loop the coefficients of all integral functions have beautiful twistor space interpretations

Box integral **Twistor space support**

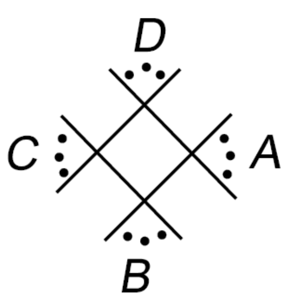
Bern, Dixon and Kosower
Britto, Cachazo and Feng

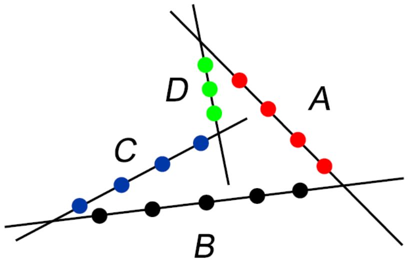
Three negative helicities





Four negative helicities





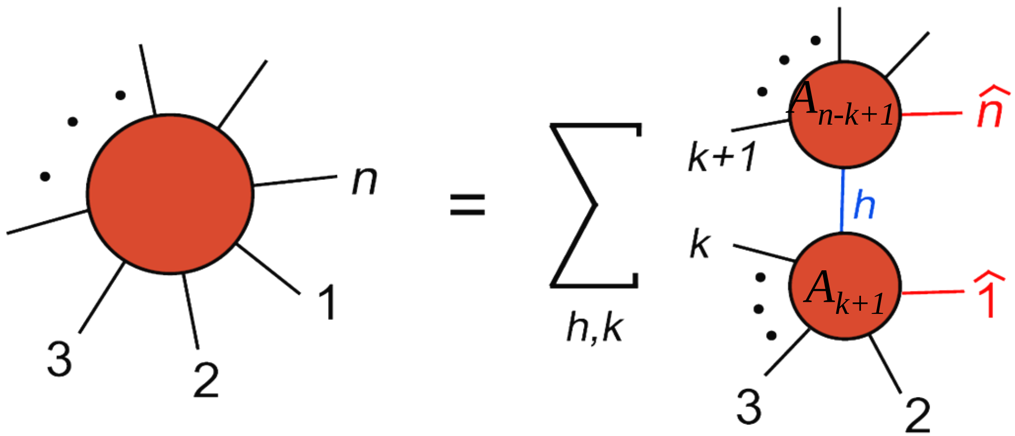
The existence of such twistor structures connected with loop-level simplicity.

Higher-loop structures **see Nima's talks**

On-Shell Recursion

A very general machinery for constructing tree level scattering amplitudes are on-shell recursion relations.

Britto, Cachazo, Feng and Witten



Building blocks are on-shell amplitudes

General replacement for tree-level Feynman diagrams

Contrast with Feynman diagram which are based on off-shell unphysical states with $p^2 \neq m^2$

Britto, Cachazo, Feng and Witten

Proof relies on so little. Power comes from generality

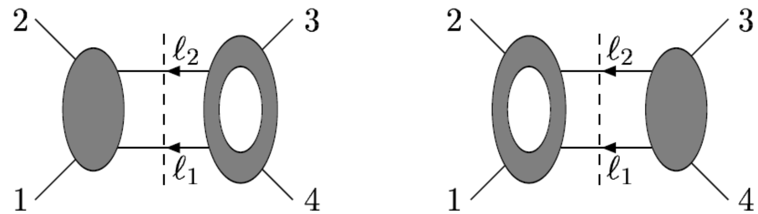
- **Cauchy's theorem**
- **Basic field theory factorization properties**
- **Applies as well to massive theories.**
- **Applies as well to gravity theories.**

Britto, Cachazo, Feng and Witten
Badger, Glover, Khoze and Svrcek
Bedford, Brandhuber, Spence, Travaglini
Cachazo and Svrcek; Benincasa, Boucher-Veronneau and Cachazo

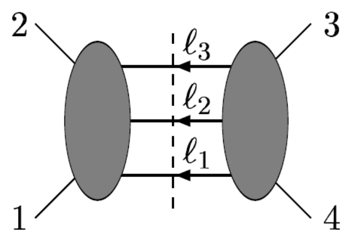
Blackboard interlude: Explicit example of on-shell recursion

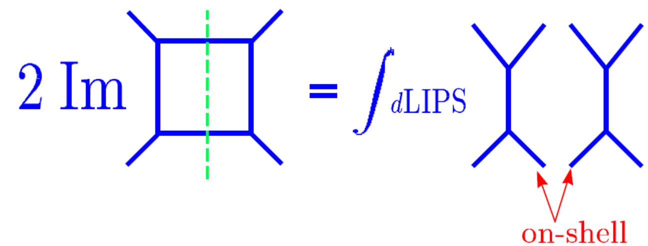
Onwards to Loops: Unitarity Method

Two-particle cut:



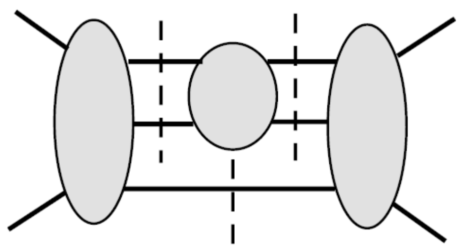
Three- particle cut:





Generalized unitarity:

Bern, Dixon and Kosower



Crucial for making high loop gravity Calculations feasible

Generalized cut interpreted as cut propagators not canceling.

A number of recent improvements to method

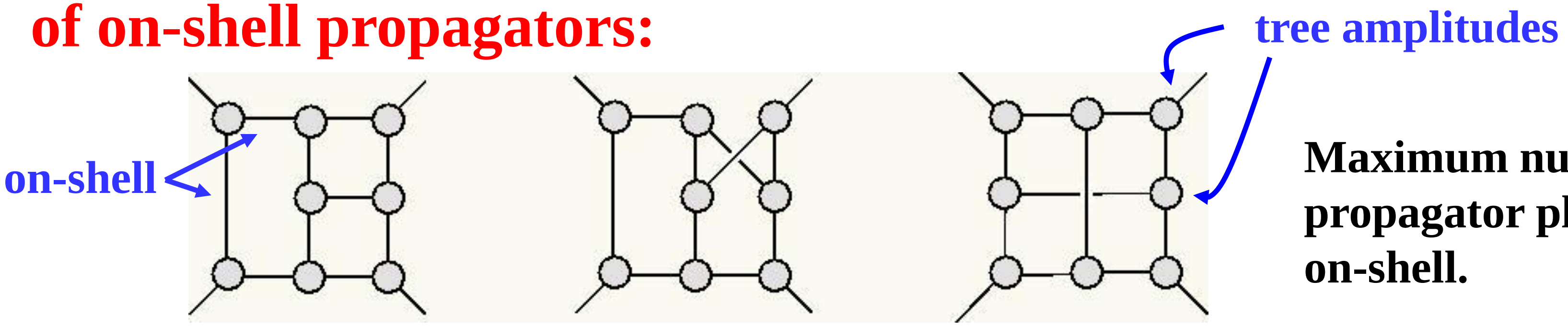
Britto, Buchbinder, Cachazo and Feng; Berger, Bern, Dixon, Forde and Kosower; Britto, Feng and Mastrolia

Method of Maximal Cuts

ZB, Carrasco, Johansson, Kosower

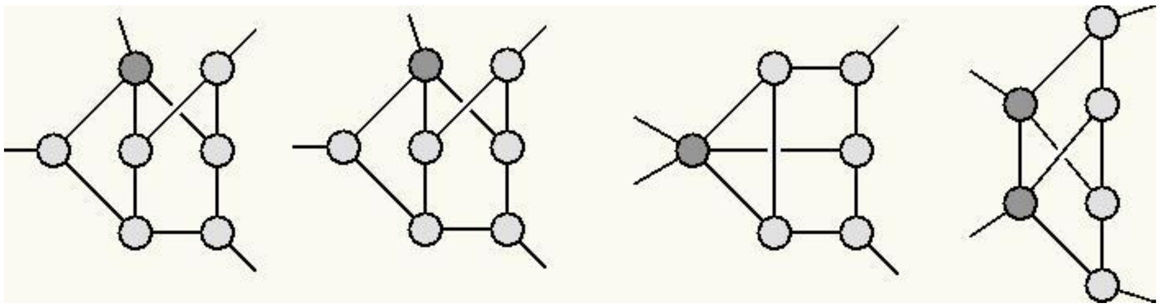
A refinement of unitarity method for constructing complete higher-loop amplitudes is "Method of Maximal Cuts". Systematic construction in *any* massless theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:



Maximum number of propagator placed on-shell.

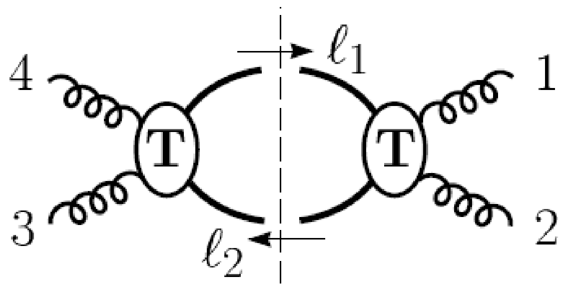
Then systematically release cut conditions to obtain contact terms:



Fewer propagators placed on-shell.

Related to leading singularities discussed by Nima.

Blackboard Interlude: Evaluate a sample cut



$$\ell_1^2 = \ell_2^2 = 0$$

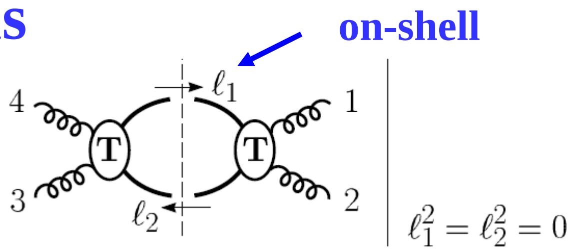
Example: *N* = 4 Loop Amplitude

Bern, Rozowsky and Yan

Consider one-loop in *N* = 4 super-Yang-Mills

1 gluon, 4 gluinos, 6 real scalars

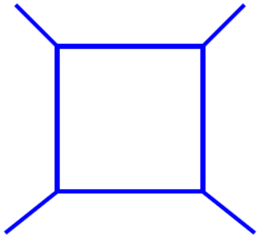
Maximal susy



The basic two-particle sewing equation

$$\sum_{N=4\ \text{states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

Applying this at one-loop gives



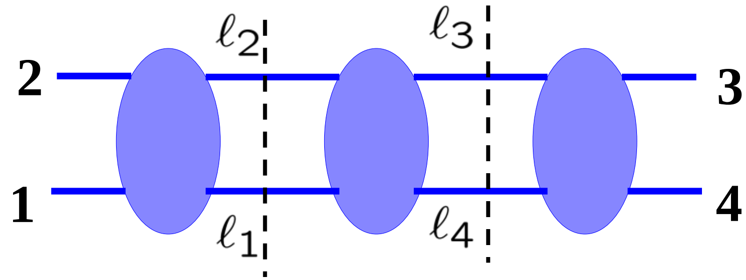
$$\mathcal{A}_4^{\text{1-loop}}(1,2,3,4) = -st A_4^{\text{tree}} \mathcal{I}_4^{\text{1-loop}}(s,t)$$

Agrees with known result of Green, Schwarz and Brink.

The two-particle cuts algebra recycles to all loop orders!

Example $N = 4$ Multi-loop Amplitude

Bern, Rozowsky and Yan



$$s = (k_1 + k_2)^2$$

$$t = (k_1 + k_4)^2$$

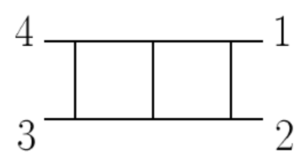
$$\sum_{N=4 \text{states}} A_4^{\text{tree}}(1,2,\ell_2,\ell_1) \times A_4^{\text{tree}}(-\ell_1,-\ell_2,\ell_3,\ell_4) \times A_4^{\text{tree}}(-\ell_4,-\ell_3,3,4)$$

$$= -s^2 t \frac{A_4^{\text{tree}}(1,2,3,4)}{(\ell_1+k_1)^2(\ell_1-\ell_4)^2(\ell_4-k_4)^2}$$

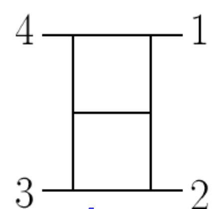
Algebra same as at one loop

Get double box integrals:

$$- st\, A_4^{\text{tree}} \Bigg\{ s$$



$$+ \; t$$

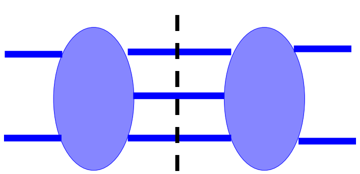


$$\Bigg\}$$

same integrals as in scalar φ^3 theory

Integrals known thanks to V. Smirnov

Also must evaluate three particle cuts:



No new contributions!

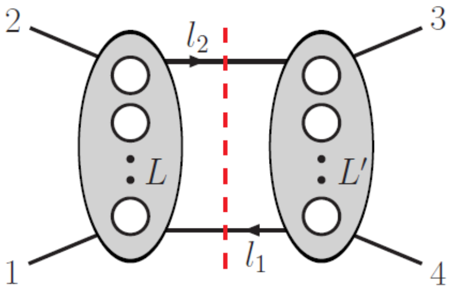
$N = 8$ supergravity is similar

Used in an all-loop resummation.
String theory matches this via AdS/CFT

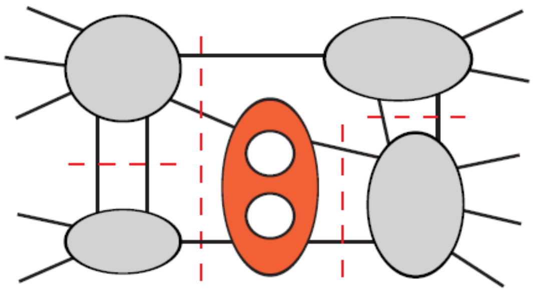
Anastasiou, Bern, Dixon, Kosower
Bern, Dixon and Smirnov
Alday and Maldacena

N = 4 sYM Cuts

Two cuts are easy to carry out in N = 4 sYM theory.



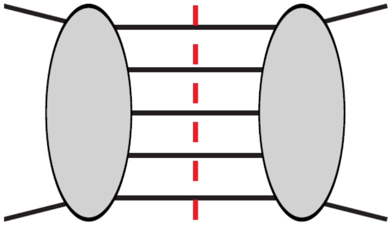
Two-particle cut



“ box cut”

This does not give everything but a substantial fraction is free easy to get this way

A more difficult cut.

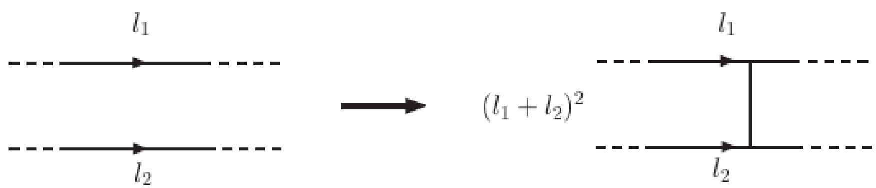


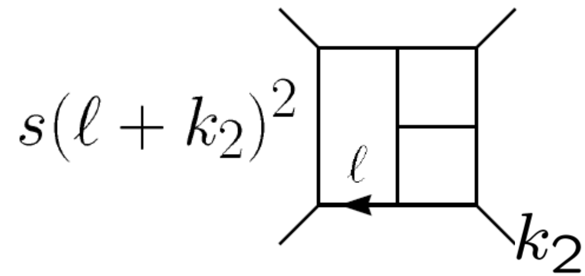
Some Pictorial Rules

Additional simplicity for maximal susy cases

- **Rung Rule (planar):**

ZB, Yan Rozowsky (1997)

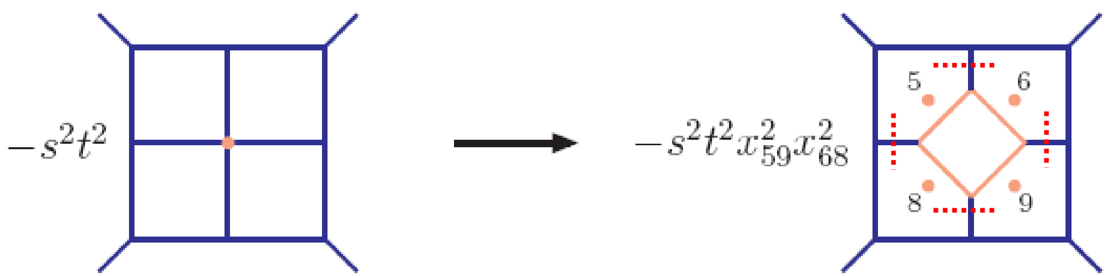




Derived from iterated 2-particle cuts

- **Box-cut substitution rule:**

ZB, Carrasco, Johansson, Kosower (2007)



***N* = 4 sYM rule**
***N* = 8 sugra similar**

Derived from generalized four-particle cuts.

If box subdiagram present, contribution easily obtained!

Similar trickery also used by Cachazo and Skinner

Summary

- Scattering amplitudes simpler than anyone imagined
  — on-shell simplicity exposed in twistor space.
- On-shell methods exploit the simplicity.
- Unitarity method gives a means for exploiting tree-level simplicity to construct higher loop amplitudes.

In next lecture we will discuss how these ideas expose a remarkable double-copy property of gravity and demonstrate non-trivial UV cancellations.

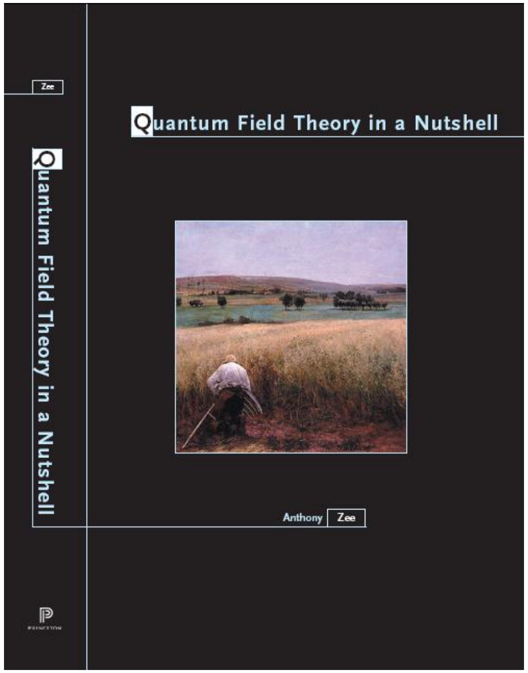
Reading List: Review Articles

- L. Dixon 1996 TASI lectures hep-ph/9601359 (trees basic ideas)
- Z. Bern, L. Dixon and D. Kosower, hep-ph/9602280 (early review on loops)
- Z. Bern, gr-qc/0206071 (early applications to gravity)
- F. Cachazo and P. Svrcek, hep-th/0504194 (twistors)
- Z. Bern, L. Dixon, D. Kosower, arXiv:0704.2798 (tools for QCD).
- Z. Bern, J.J.M. Carrasco, H. Johansson, arXiv:0902.3765 (UV properties of gravity)
- R. Woodard, arXiv:0907.4238 (gravity)
- L. Dixon , arXiv:1005.2703 (UV properties of gravity)

Further Reading

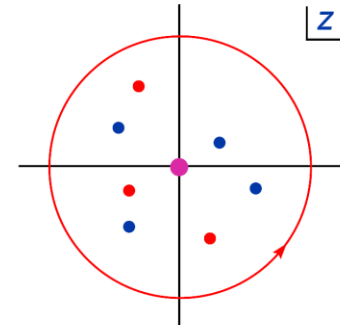
Hermann Nicolai, ***Physics Viewpoint*****, "Vanquishing Infinity"**
http://physics.aps.org/articles/v2/70

Anthony Zee, ***Quantum Field Theory in a Nutshell*****,**
2nd Edition is first textbook to contain modern formulation of scattering and commentary on new developments. 4 new chapters.



Extra Transparancies

Large z behavior

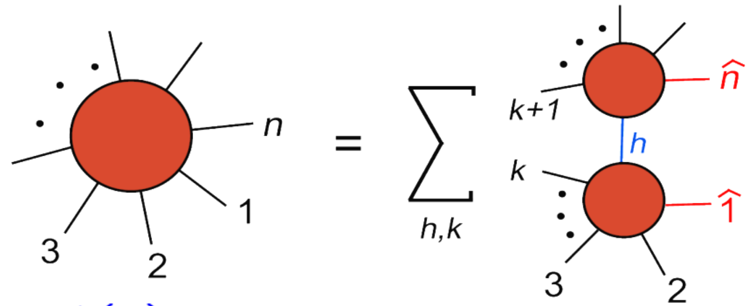


Consider amplitude under complex deformation of an amplitude. Proof of on-shell recursion relies on good behavior at large z.

$$p_1^\mu(z) = p_1^\mu - \frac{z}{2}\langle 1^- |\gamma^\mu| n^-\rangle \qquad p_n^\mu(z) = p_n^\mu + \frac{z}{2}\langle 1^- |\gamma^\mu| n^-\rangle \qquad (p_i^\mu(z))^2 = 0$$

complex momenta

A(z) is amplitude with shifted momenta



If $A(z) \to 0, \quad z \to \infty$

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \text{Res}_\alpha \frac{A(z)}{z}$$

$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

Poles in z come from kinematic poles in amplitude.

Sum over residues gives the on-shell recursion relation

Remarkably, gravity is as well behave at $z \to \infty$ as gauge theory
Will play a crucial role in understanding how the high-energy behavior in quantum gravity can be better than people thought.

A Remarkable Twistor String Formula

The following formula encapsulates the entire tree-level S-matrix of *N* = 4 super-Yang-Mills:

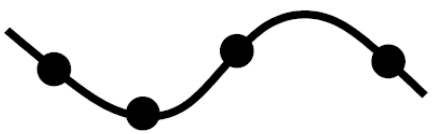
Witten
Roiban, Spradlin and Volovich

Integral over the Moduli and curves

$$A_n = i(2\pi)^4 g_{\mathrm{YM}}^{n-2} \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2\left(\lambda_i^\alpha - \xi_i P_i^\alpha\right) \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{\alpha}}\right) \delta^4\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA}\right)$$

$$P_i^\alpha = \sum_{k=0}^{d} a_k^\alpha \sigma_i^k$$

Degree *d* polynomial in the moduli σ_i



Strange formula from Feynman diagram viewpoint.

But it's true: impressive checks by Roiban, Spradlin and Volovich

Recently this formula has been connected to generalized unitarity using the global residue theorem. Spradlin and Volovich