SOME COEFFICIENTS HAVE BEEN WORKED OUT CAREFULLY, BUT SOME ARE PURELY IMPRESSIONISTIC.

1. Supersymmetric Lagrangians

We can start from theories of scalars and fermions. Gauge fields will be introduced later. To write supersymmetric Lagrangians we will use chiral superfields. These are defined as a representation of supersymmetry that satisfies

$$[\overline{Q}_{\dot{\alpha}}, \phi(x)] = 0 \ . \tag{1.1}$$

The representation is therefore constructed by acting with Q_α and ∂_μ. This is organized in a superfield that satisfies $\overline{D}_{\dot{\alpha}}\Phi = 0$ and Φ is called a chiral superfield. The solution is

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \ , \tag{1.2}$$

with $y^\mu = x^\mu + \theta\sigma^\mu\overline{\theta}$. Note that any function of chiral superfields, $W(\Phi_i)$, is by itself a chiral superfield.

The main property which will be relevant for us is that we can compute the SUSY transformations of the F-term and get

$$[Q_\alpha, F] = 0 \ , \qquad [\overline{Q}_{\dot{\alpha}}, F] \sim \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^\alpha \ . \tag{1.3}$$

We see that the $\overline{Q}$ variation is a total derivative while the Q variation is zero.

Similarly, let us consider a general real superfield, R

$$R = \cdots + \overline{\theta}^2\theta\lambda + \theta^2\overline{\theta\lambda} + \theta^4 D \ . \tag{1.4}$$

We can study the SUSY variations of the top component, D, and get

$$[Q_\alpha, D] \sim \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\overline{\lambda}^{\dot{\alpha}} \ , \qquad [\overline{Q}_{\dot{\alpha}}, D] \sim \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\lambda^\alpha \ . \tag{1.5}$$

We see that the variation of the top component is again a total derivative.

This gives us a general prescription for constructing supersymmetric Lagrangians

$$\mathcal{L} = D + F + \overline{F} \ . \tag{1.6}$$

The variation of each term is a total derivative and thus the action $S = \int d^4x\mathcal{L}$ is SUSY-invariant. F is the θ^2 component of a function of the superfields which is called the superpotential $W(\Phi)$ (and it is chiral, thus, independent of the complex conjugates). D is the top component of a real function of the superfields and their complex conjugates $K(\Phi^i, \overline{\Phi}^i)$. This function is called the Kähler potential. A neat way to extract the top components F, D is to write Grassman integrals $\int d^2\theta$, $\int d^4\theta$ respectively. Thus we arrive at the usual way supersymmetric Lagrangians are presented

$$\mathcal{L} = \int d^4\theta K(\Phi^i, \overline{\Phi}^i) + \left(\int d^2\theta W(\Phi^i) + c.c \right) . \tag{1.7}$$

Having developed this formalism, let us give some basic examples. With a single chiral superfield $K = \Phi^\dagger\Phi$, $W = 0$ corresponds to a free theory of one fermion and a complex boson. The Lagrangian in components is

$$\mathcal{L} = \phi^\dagger \Box\, \phi + i\partial_\mu\psi\sigma^\mu\overline{\psi} + F^\dagger F . \tag{1.8}$$

We see that F is not propagating, its equation of motion implies $F = 0$ and thus it can be removed from (1.8). This free theory is the simplest supersymmetric theory. The number of bosons and fermions is the same; this is a general consequence of supersymmetry.

The free theory can be generalized to describe the supersymmetric sigma model. We take any number N of chiral superfields Φ^i and a general real function $K(\Phi^i, \overline{\Phi}^i)$. The superpotential vanishes. This theory takes the form (after setting to zero the auxiliary fields)

$$\mathcal{L} = -g_{i\bar{j}}(\phi, \phi^\dagger)\partial_\mu\phi^{\dagger\bar{j}}\partial^\mu\phi^i + ig_{i\bar{j}}\partial_\mu\psi^i\sigma^\mu\overline{\psi}^{\bar{j}} + ig_{i\bar{j}}i\Gamma^i_{kl}(\partial_\mu\phi^k)\psi^l\sigma^\mu\overline{\psi}^{\bar{j}} + \frac{1}{4}R_{i\bar{j}k\bar{l}}\psi^i\psi^k\overline{\psi}^{\bar{j}}\overline{\psi}^{\bar{l}} . \tag{1.9}$$

The first term is the conventional kinetic term in the non-linear σ-model. Physically it describes a theory whose target space is an N complex dimensional manifold with metric $g_{i\bar{j}}$. The metric is related to the Kähler potential via

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K . \tag{1.10}$$

This means that the target space is a Kähler manifold. Γ and R are the connection and Riemann tensor of this geometry, respectively.

Let us explain a little more some of the geometric concepts involved. The metric remains invariant under redefinitions of the Kähler function of the form

$$K'(\Phi^i, \overline{\Phi}^i) = K(\Phi^i, \overline{\Phi}^i) + F(\Phi^i) + \overline{F}(\Phi^{\bar{i}}) \ . \tag{1.11}$$

This can be seen from the direct definition of $g_{i\bar{j}}$ and also from the superspace Lagrangian (1.7), since $\int d^4\theta$ acting on chiral or antichirals superfields gives a total derivative.

The bosons ϕ^i should be thought of as coordinates on the target space manifold. This is natural because the physical theory is invariant under any field redefinition of the form $\Phi'^i = f^i(\Phi^j)$. The metric transforms in the usual tensorial way. However, fermions transform as $\psi'^i = \frac{\partial \phi'^i}{\partial \phi_k}\psi^k$. This suggests that the fermions should be identified with tangent vectors on the target space.

An example of an interesting sigma-model is obtained if we consider the target space to be the two-sphere $S^2 = \mathbb{CP}^1$. The theory is $K = f_\pi^2 \log(1+|\Phi|^2)$ where f_π^2 is proportional to the radius-squared of the sphere. Indeed, using the formula (1.10) we get the Kähler metric $g_{\Phi\overline{\Phi}} = \frac{f_\pi^2}{(1+|\phi|^2)^2}$. This is the familiar round metric on the sphere. The distance to $\Phi \sim \infty$ is finite with this metric. To describe the region of infinity we need to perform a change of variables $\Phi \to 1/\Phi$. This takes the Kähler metric to itself up to a Kähler transformation proportional to $\log(\Phi) + \log(\overline{\Phi})$. (As we explained, this Kähler transformation does not affect the physical theory.) Finally, one can check that this theory has the full $SU(2)$ isometry of the sphere. SUSY vacua are field configurations corresponding to points on the sphere. The global $SU(2)$ symmetry is then broken to a $U(1)$ symmetry.

Let us now also discuss theories with nonzero superpotential function. The simplest example is the free Kähler potential $K = \Phi^\dagger\Phi$ with superpotential $W = \frac{1}{2}m\Phi^2$. This gives a mass m to the free complex boson and Weyl fermion in Φ. More generally, adding a superpotential $W(\Phi^i)$ to the sigma model with Kähler potential $K(\Phi^i, \overline{\Phi}^{\bar{j}})$ adds to the theory (1.9) a scalar potential and Yukawa-type interactions

$$\mathcal{L}' = (1.9) - g^{i\bar{j}}\partial_i W \partial_{\bar{j}}\overline{W} - \left(\partial_i\partial_j W \psi^i\psi^j + c.c.\right) \ . \tag{1.12}$$

Again, we have solved the equations of motion of the auxiliary fields F^i and their complex conjugates. Supersymmetric field configurations should have zero energy. Since the metric is positive definite, this means that $\partial_i W = 0$ for all i.

We should now discuss how to add gauge interactions. The gauge potential A_μ is embedded in the $\theta\overline{\theta}$ component of a real superfield V. Gauge transformations are achieved by demanding invariance under

$$V \to V - i(\Lambda - \overline{\Lambda}) \ , \tag{1.13}$$

with chiral Λ. It is possible to define the chiral superfield $W_\alpha = \frac{-1}{4}\overline{D}^2 D_\alpha V$. It contains only the gauge invariant pieces from the superfield V. The kinetic term for the gauge field is written as

$$\mathcal{L}_{gaugekinetic} = \int d^2\theta \frac{1}{4g^2} W_\alpha^2 + c.c = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{i}{g^2}\partial_\mu \lambda \sigma^\mu \overline{\lambda} + \frac{1}{2g^2} D^2 \ . \tag{1.14}$$

D is a real auxiliary field that should be integrated out. In this trivial theory it is just set to zero by its equations of motion.

Adding charged matter is achieved by letting the superfields transform under gauge transformations. For example, a chiral superfield of charge $+1$ transforms like $\Phi \to e^{i\Lambda}\Phi$. Then, a term like $\Phi^\dagger e^V \Phi$ in the Kähler potential is gauge invariant. This formalism carries over to the non-abelian case with only few technical complications.

The simplest $U(1)$ gauge theory has particles of charge ± 1. This is called supersymmetric QED (SQED). Its Lagrangian takes the form

$$\mathcal{L} = \int d^2\theta \frac{1}{4g^2} W_\alpha^2 + c.c. + \int d^4\theta \Phi_+^\dagger e^V \Phi_+ + \Phi_-^\dagger e^{-V} \Phi_- \ . \tag{1.15}$$

To see how this looks like in components we can fix a convenient gauge in which $V| = V|_\theta = V|_{\overline{\theta}} = V|_{\theta^2} = V|_{\overline{\theta}^2} = 0$. This is called WZ gauge. (The remaining gauge transformations are the ordinary real gauge transformations.) We also solve the equations of motion of the auxiliary D-term (which in this case are nontrivial). Then, the scalar potential takes the form

$$V_D = \frac{g^2}{8}\left(|\phi_+|^2 - |\phi_-|^2\right)^2 \ . \tag{1.16}$$

In addition the Lagrangian (1.15) leads to the usual gauge interaction terms and some Yukawa-type interactions of the form $\frac{-i}{\sqrt{2}}\left(\phi_+ \overline{\psi}_+ \overline{\lambda} - \phi_+^\dagger \psi_+ \lambda - (+ \leftrightarrow -)\right)$. The space of supersymmetric field configurations is given by $|\phi_+| = |\phi_-|$. We can use the ordinary gauge transformations to set ϕ_- real. Hence, the moduli space is really one complex dimensional, parameterized by arbitrary $\phi_+ = re^{i\theta}$.

For the special case of $U(1)$ gauge theories we can add a special gauge-invariant term to the Lagrangian of the form $\int d^4\theta \xi V$. This is called the Fayet-Iliopoulos term (FI-term). (Of course, ξ is real.) It is gauge invariant because the $\int d^4\theta$ of a chiral superfield is a total derivative. The effect of this term is just to change the scalar potential to

$$V_D = \frac{g^2}{8}\left(|\phi_+|^2 - |\phi_-|^2 + \xi\right)^2 \ . \tag{1.17}$$

The space of SUSY vacua (moduli space) thus is still parameterized by a one complex dimensional coordinate $\phi_+ = re^{i\theta}$ but the metric would be different than in the case with $\xi = 0$.

Let us say several more words about SQED. The theory is IR free and so is described by the photon and the chiral superfields $\Phi_\pm$ at low energies. It breaks down near the Landau pole, where it needs a UV completion to make sense. The supersymmetric moduli spaces we have discussed are not lifted by any order in perturbation theory, and thus remain good SUSY vacua in the full quantum theory.[1] This theory therefore does not break SUSY spontaneously.

2. Examples of SUSY-Breaking Theories

SUSY-breaking theories are theories in which there are vacua with nonzero energy. Similarly to the Goldstone phenomenon for ordinary spontaneously broken symmetries, broken supersymmetry leads always to a massless fermion, the Goldstino.

The simplest SUSY breaking theory is just one chiral superfield Φ with a linear term in the superpotential (sometimes referred to as the "Polonyi model")

$$\mathcal{K} = \int d^4\theta \Phi^\dagger \Phi + \left(\int d^2\theta f \Phi + c.c. \right) \ . \tag{2.1}$$

From our previous discussion we see that the scalar potential in this theory is simply a constant $V = |f|^2$. Thus, the spectrum consists of a massless fermion ψ (this is the Goldstino in this theory) and the scalar in this theory, ϕ is massless too. So there are infinitely many inequivalent vacua with nonzero energy and since the theory is free they are not lifted. Thus this theory is not really interesting.

[1] This follows from the non-renormalization theorems in SUSY, we may get to it later again.

We can modify it slightly by making the Kähler potential non-canonical

$$\mathcal{K} = \int d^4\theta \left(\Phi^\dagger \Phi - \frac{1}{4M^2} \Phi^2 (\Phi^\dagger)^2 \right) + \left(\int d^2\theta f \Phi + c.c. \right) \ , \tag{2.2}$$

with M some high cutoff scale. Now this theory is more interesting. The scalar potential is

$$V = \frac{1}{1 - \frac{1}{M^2}\phi^\dagger \phi} |f|^2 = |f^2| \left(1 + \frac{|\phi|^2}{M^2} + \mathcal{O}\left(\frac{1}{M^4} \right) \right) \ . \tag{2.3}$$

Now we have a good vacuum at $\phi = 0$ and there is no more infinite degeneracy as the boson ϕ has mass $|f|^2/M^2$ in the vacuum at the origin. The fermion ψ is still massless and plays the role of the Goldstino. This situation is very typical: scalar fields which are not Goldstone bosons are massive and some fermions may remain massless due to various reasons (Goldstino, 't-Hooft matching).

One may complain that (2.2) is not renormalizable. The O'Raifeartaigh model is the simplest renormalizable model that breaks SUSY spontaneously. The model has three chiral superfields, the Kähler potential is canonical and the superpotential is

$$W = X\Phi^2 - mN\Phi - fX \ . \tag{2.4}$$

The conditions for the existence of a SUSY vacuum would be $\phi = 0, \phi^2 = f, mN = 2x\phi$. Clearly there is no solution. There is a SUSY breaking vacuum at $\Phi = N = 0$. The vacuum energy is $V = |f|^2$. This vacuum has no tachyons for $2f < m^2$. However, since x always appears together with ϕ in the scalar potential, there is no energy cost in changing x. Thus, our vacuum, again, is part of a one-complex dimensional family of degenerate vacua. (This is a very general property of such theories.) Unlike the previous theory (which was free) here radiative corrections lift this pseudo-modulus space and give a positive mass squared for x at $x = 0$:

$$V^{1-loop} \sim \frac{1}{16\pi^2} \frac{|f|^2}{m^2} |x|^2 + \dots \ . \tag{2.5}$$

Thus, the SUSY-breaking vacuum at $x = 0$ is gapped (besides the goldstino ψ_x which is massless of course) and this constitutes the simplest construction of a controllable SUSY-breaking model.

Homework: Analyze the theory (at tree-level) for $2f > m^2$. Is there an infinite degeneracy of vacua?

Let us now discuss some SUSY-breaking models based on the FI-term. The simplest example is a pure $U(1)$ gauge theory with a FI-term

$$\mathcal{L} = \int d^2\theta \frac{1}{4g^2} W_\alpha^2 + c.c. + \int d^4\theta \xi V \ . \tag{2.6}$$

The vacuum is $V = g^2\xi^2/8$. The gauge field and gaugino are massless, the gaugino is also the Goldstino in this theory. Again, this is a boring free theory.

The model can be complicated a little to exhibit more interesting phenomena. Consider two chiral $\Phi_\pm$ superfields with $U(1)$ charges ± 1. We also include a mass term for them and a FI-term. This is a slight generalization of SQED. Thus, the theory is

$$\mathcal{L} = \int d^2\theta \frac{1}{4g^2} W_\alpha^2 + c.c. + \int d^4\theta \left(\Phi_+^\dagger e^V \Phi_+ + \Phi_-^\dagger e^{-V} \Phi_- + \xi V \right) + \left(\int d^2\theta m \Phi_+ \Phi_- + c.c \right) \ . \tag{2.7}$$

The scalar potential is

$$V = \frac{g^2}{8} \left(|\phi_+|^2 - |\phi_-|^2 + \xi \right)^2 + m^2 \left(|\phi_+|^2 + |\phi_-|^2 \right)) \ . \tag{2.8}$$

This potential does not admit any SUSY preserving vacua. Indeed, it is a sum of positive terms that cannot be set to zero simultaneously.

The theory has several phases, depending on m^2/ξ. For m^2/ξ smaller than some critical value κ_c the theory Higgses the gauge symmetry. For m^2/ξ larger than κ_c the theory breaks SUSY but without Higgsing the $U(1)$ symmetry. (The transition at κ_c is of second order.)

Homework: Find κ_c. For the more advanced: This theory has an R-symmetry. Is it broken in the vacuum?

3. Brief Introduction to Strongly Coupled Theories and SUSY-Breaking

So far we have discussed mostly classical phenomena, and presented theories that break SUSY. If our goal is to use SUSY for describing something in the real world, it must be broken (probably spontaneously). Playing with tree-level theories is educational but phenomenologically unmotivated, since we need to introduce the SUSY breaking scale by hand.

Unlike other symmetries, there is a very strong non-renormalization theorem about supersymmetry. The theorem states that *Supersymmetric vacua cannot be lifted by radiative corrections at any order.* This means that if there is some SUSY vacuum in the classical theory, $\phi_i^{(0)}$, then it will not disappear once we compute the effective potential in perturbation theory.

This has led Witten to observe that therefore there are two options:

1. The theory has no SUSY vacua classically (in this case we say that it breaks SUSY at tree-level).
2. The theory does have SUSY vacua classically but they are lifted by non-perturbative effects.

Of course, option 1. is unappealing, since we would need to introduce dimensionful parameters by hand. On the other hand, option 2 is very interesting since it means that the SUSY-breaking scale (i.e. the fourth root of the vacuum energy density, loosely denoted as $\sqrt{F}$) is related to the UV-cutoff of the theory as

$$\sqrt{F} \sim \Lambda_{UV} e^{-\frac{8\pi^2}{g^2}} \ . \tag{3.1}$$

This scale is exponentially smaller than Λ_{UV}.

Thus, this mechanism can potentially explain why SUSY-breaking is happening at a low scale without putting unexplained small parameters by hand in the Lagrangian! The hope is that it will shed light on the origin of the scale of the Standard Model (100 GeV).

Hence, we should really be studying IR-strong field theories that can lead to SUSY-breaking. This is a vast subject that will not be covered here, see (ref) for reviews.

Instead, we will comment that, roughly speaking, these theories UV-free theories fall into two classes

1. Theories in which powerful tools such as holomorphy and Seiberg-duality are useful, and we can calculate the vacuum energy, masses of some particles etc.
2. Theories for which there are strong (but usually indirect) arguments that SUSY is broken but we cannot say much about the properties of the vacuum.

The first type of theories is referred to as "calculable theories." We will not discuss theories of the second type (incalculable theories) here. Calculable theories arise when the physics of the strong coupling scale Λ can be integrated out and we are left with a bunch of light moduli fields, with some Kähler potential, global symmetries, and perhaps IR-free gauge fields.

This is reminiscent of pion physics, which occurs at scales below the strong coupling scale and leads to many precise predictions.

Indeed, calculable dynamical theories lead at low energies to generalized O'Raifeartaigh models (2.4), interesting sigma models (with and without gauged global symmetries), possible effective FI-terms etc. Therefore, studying such classical theories is useful and can very often explain general properties of interesting calculable dynamical models.

The next section begins with a deeper study of the structure of supersymmetric theories. We will then use our results to shed light on the possible dynamics of various models.

4. The Supercurrent

Supersymmetric theories in 4d with $\mathcal{N}=1$ supersymmetry have a conserved energy momentum tensor $T_{\mu\nu}$ as well as a supercurrent $S_{\mu\alpha}$

$$\partial^\mu S_{\mu\alpha} = 0 \ , \qquad \partial^\mu T_{\mu\nu} = 0 \ . \tag{4.1}$$

In theories with symmetries all the operators furnish representations of the symmetries. In particular, in supersymmetric theories the operators should sit in supersymmetric multiplets, or equivalently, superfields. We can get a hint of what is going to happen from the following observations:

1. The SUSY algebra is
$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} \sim \sigma^\mu_{\alpha\dot\alpha} P_\mu \ . \tag{4.2}$$
This can be written as $\{Q_\alpha, \int d^3x \overline{S}_{0\dot\alpha}\} \sim \sigma^\mu_{\alpha\dot\alpha} \int d^3x T_{0\mu}$. From here we see that
$$\{Q_\alpha, \overline{S}_{\nu\dot\alpha}\} \sim \sigma^\mu_{\alpha\dot\alpha} T_{\mu\nu} + \cdots \ . \tag{4.3}$$
Here $\cdots$ stand for terms which are space derivatives when we set ν to zero. In addition these terms must be conserved when we act with ∂^ν. The conclusion is that the supercurrent and energy momentum must sit in the same supermultiplet.
2. We expect that the energy momentum tensor is the highest spin component in the multiplet. If this were not the case, there would be problems in coupling the theory to supergravity (a topic we will get back to later).

From these we conclude that the superfield must be a vector superfield with the following rough structure

$$\mathcal{J}_\mu = j_\mu + \theta^\alpha \left(S_{\mu\alpha} + \cdots\right) + \overline{\theta}_{\dot\alpha}\left(\overline{S}^{\dot\alpha}_\mu + \cdots\right) + \left(\theta\sigma^\nu\overline{\theta}\right)\left(2T_{\nu\mu} + \cdots\right) + \cdots \ . \tag{4.4}$$

The vector operator in the bottom component is generally not conserved. There is not a unique way to fix the full structure (i.e. to fix the $\cdots$).

The simplest solution is called the Ferrara-Zumino multiplet. It is given by the defining equations

$$-2\sigma^{\mu}_{\alpha\dot{\alpha}}\overline{D}^{\dot{\alpha}}\mathcal{J}^{FZ}_{\mu} = D_{\alpha}X \ , \qquad \overline{D}X = 0 \ . \tag{4.5}$$

In addition, $\mathcal{J}^{FZ}$ is real. The solution in components takes the form

$$\begin{aligned}\mathcal{J}^{FZ}_{\mu} &= j_{\mu} + \left(\theta^{\alpha}\left(S_{\mu\alpha} + \frac{1}{3}(\sigma_{\mu}\overline{\sigma}^{\rho}S_{\rho})_{\alpha}\right) + c.c.\right) + \frac{i}{2}\theta^2\partial_{\mu}\overline{x} - \frac{i}{2}\overline{\theta}^2\partial_{\mu}x \\ &+ \left(\theta\sigma^{\nu}\overline{\theta}\right)\left(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}T - \frac{1}{4}\epsilon_{\nu\mu\rho\sigma}\partial^{[\rho}j^{\sigma]}\right) + \mathcal{O}(\theta^3) \ .\end{aligned} \tag{4.6}$$

The terms with three and four θs do not contain new operators, only derivatives of those we have already displayed. The expression for X (in terms of the usual y coordinate) is

$$X = x + \sqrt{2}\theta\psi + \theta^2 F \ , \qquad \psi = \frac{\sqrt{2}}{3}\sigma^{\mu}\alpha\dot{\alpha}\overline{S}^{\dot{\alpha}}_{\mu} \ , \quad F = \frac{2}{3}T + i\partial_{\nu}j^{\nu} \ . \tag{4.7}$$

This expression can be used to find the current algebra relation (4.3).[2] The number of bosonic operators in this multiplet is $12 = 2(x) + 4(j_{\mu}) + 6(T_{\mu\nu})$ and the same for the fermions.

Given that this is the minimal representation, there is an obvious question to ask

Does this multiplet exist for all the supersymmetric field theories we have discussed so far?

We will answer this question by looking at examples. Let us start from a free field theory of one chiral superfield with canonical Kähler potential and vanishing superpotential

$$\mathcal{L} = \int d^4\theta\Phi^{\dagger}\Phi \ . \tag{4.8}$$

[2] Roughly, it takes the form

$$\{Q_{\alpha}, \overline{S}_{\mu\dot{\alpha}}\} = \sigma^{\nu}_{\alpha\dot{\alpha}}\left(2T_{\mu\nu} + i\eta_{\mu\nu}\partial_{\nu}j^{\nu} - i\partial_{\nu}j_{\mu} - \frac{1}{4}\epsilon_{\nu\mu\rho\sigma}\partial^{[\rho}j^{\sigma]}\right)$$

It can be checked that the algebra of charges is satisfied and the conservation equations are consistent.

Then the equation of motion can be nicely written in superspace as[3]

$$\overline{D}^2\Phi^\dagger = 0 \ . \tag{4.9}$$

Using this we can easily check that the multiplet

$$\mathcal{J}^{FZ}_{\alpha\dot{\alpha}} = 2D_\alpha\Phi\overline{D}_{\dot{\alpha}}\Phi^\dagger - \frac{2}{3}[D_\alpha, \overline{D}_{\dot{\alpha}}]\Phi^\dagger\Phi \ , \tag{4.10}$$

satisfies

$$\overline{D}^{\dot{\alpha}}\mathcal{J}^{FZ}_{\alpha\dot{\alpha}} = 0 \ . \tag{4.11}$$

(In the above we have adopted the convention $-2\sigma^\mu_{\alpha\dot{\alpha}}l_\mu = l_{\alpha\dot{\alpha}}$ for switching between the bi-spinor and the vector notation.) The fact that $X = 0$ for this theory means that the energy-momentum tensor is traceless and the whole theory is conformal, which is indeed reasonable for the free field theory.

More generally, a sigma model with Kähler potential K and superpotential W admit the multiplet

$$\mathcal{J}^{FZ}_{\alpha\dot{\alpha}} = 2g_{i\bar{j}}D_\alpha\Phi^i\overline{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}} - \frac{2}{3}[D_\alpha, \overline{D}_{\dot{\alpha}}]K \ , \tag{4.12}$$

and it satisfies the FZ equation with $X = 4W - \frac{1}{3}\overline{D}^2K$.

We will get back to this formula later. Let us now find the FZ-multiplet for the FI-model. We may consider, for simplicity, the pure $U(1)$ FI-term (2.6). The equation of motion gives $D^\alpha W_\alpha = -\xi$ (and we also have the usual Bianchi identity $D^\alpha W_\alpha = \overline{D}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}$). Using this one can check that

$$\mathcal{J}^{FZ}_{\alpha\dot{\alpha}} = \frac{-4}{g^2}W_\alpha\overline{W}_{\dot{\alpha}} - \frac{2}{3}\xi[D_\alpha, \overline{D}_{\dot{\alpha}}]V \tag{4.13}$$

satisfies the FZ equation with $X \sim \xi\overline{D}^2V$. The expression in the presence of matter fields can be easily obtained as well.

Now we come to our original question, which concerned itself with the conditions for the existence of the FZ-multiplet.

For an operator to be a legitimate "local operator" in the theory it must satisfy the following three conditions

[3] To see this, allow a general variation of $\delta\Phi$ and impose that the resulting action is stationary. We get $\int d^4\theta\Phi^\dagger\delta\Phi + c.c. = 0$. We can then write this as $\sim \int d^2\theta\overline{D}^2\Phi^\dagger\delta\Phi + c.c.$ which vanishes for a generic $\delta\Phi$ if and only if $\overline{D}^2\Phi^\dagger = 0$.

1. It can be expressed locally (with finitely many derivatives) in terms of the fields of the theory (once a microscopic description is provided).
2. It is gauge invariant.
3. It is a globally well-defined operator in the field space of the theory (we will see an example of this in the following).

The question is whether the FZ-multiplet is a well-defined operator in the theory. Clearly, for the FI-model it is not gauge invariant under the full supersymmetric group of gauge transformations (1.13). Does this mean that there is no energy momentum tensor in this theory? and a supercurrent? Under a gauge transformation the multiplet transforms as

$$\delta \mathcal{J}^{FZ}_{\alpha\dot{\alpha}} = i\frac{2}{3}\xi[D_\alpha, \overline{D}_{\dot{\alpha}}](\Lambda - \overline{\Lambda}) \sim \xi\partial_{\alpha\dot{\alpha}}(\Lambda + \overline{\Lambda}) \ , \qquad \delta X \sim i\xi\overline{D}^2\Lambda \ . \tag{4.14}$$

We see that the FZ-multiplet is not gauge invariant. Let us study this non-gauge invariance in more detail. It can be checked that the transformation (4.14) acts on the energy momentum tensor and on the supercurrent in the following way

$$\delta T_{\mu\nu} \sim \xi \left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right) \mathrm{Im}\,\Lambda\big| \ , \qquad \delta S_{\mu\alpha} \sim \xi(\sigma_{\mu\nu})_\alpha^{\ \beta}\partial^\nu(\psi_\Lambda)_\beta \ . \tag{4.15}$$

Here $\sigma_{\mu\nu}$ is the usual antisymmetric combination $\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu)$.

This kind of transformations is very special. They do not affect the charges Q_α, P_μ (because both the transformation of energy momentum and the supercurrent reduce to space derivatives when $\mu = 0$) and the conservation equation is not affected either. Such special transformations go under the name "improvement terms."

We see that the situation is somewhat subtle; the operator $\mathcal{J}^{FZ}$ does not strictly exist in the theory, because it is not gauge invariant. However, the theory is not really sick since the charges and the SUSY algebra do exist. This has many consequences which we will get to soon.

Let us now consider the multiplet for sigma model (4.12). We see that it is not invariant under Kähler transformations. Moreover, the structure is identical to (4.14),(4.15). The energy momentum tensor and the supercurrent only change by improvement terms.

However, unlike gauge transformations, Kähler transformations are not sacred. It does not matter much if the multiplet transforms under Kähler transformations. There is one exception in which Kähler transformations become important. This is when the target space manifold, $\mathcal{M}$, needs to be covered with several patches and Kähler transformations are necessary when we switch between the patches. We have already seen such an example before, the $\mathbb{CP}^1$ sigma model.

Let us understand the mathematical condition more precisely. The metric tensor $g_{i\bar{j}}$ can also be thought of as a two-form

$$\omega \sim i g_{i\bar{j}} d\Phi^i \wedge d\overline{\Phi}^{\bar{j}} \ . \tag{4.16}$$

This two-form is closed due to (1.10). Therefore, on every single patch, we can find a one-form $\mathcal{A}$ such that $d\mathcal{A} = \omega$. It is given by $\mathcal{A} \sim i\partial_i K d\Phi^i + c.c.$.

Note that the bottom component of the FZ-multiplet (4.12) has some fermionic terms and a bosonic piece proportional to $i\frac{\partial K}{\partial \Phi^i}\partial_\mu \Phi^i + c.c.$. This is the pull back to space time of the one form $\mathcal{A}$. The FZ multiplet is well-defined only if the bottom component is well-defined, and this holds if the one-form $\mathcal{A}$ indeed exists. This means that ω vanishes in $H^2(\mathcal{M})$.

We conclude that the vanishing of (4.16) in $H^2(\mathcal{M})$ is a necessary condition for the existence of the FZ-multiplet. In other words, the Kähler form must be exact. An immediate corollary follows from the fact that the volume form of compact Kähler manifolds (such as the $\mathbb{CP}^1$) is given by $\omega^{dim(\mathcal{M})/2}$. Since the volume form cannot be exact, it is clear that ω cannot be exact for compact manifolds. Hence, if the target space is compact it has no well-defined FZ-multiplet.

If (4.16) is not exact, the FZ-multiplet it is not a function on the target space of the theory. Different "observers" can measure the energy momentum tensor relative to two different patches (which intersect) and report different values (however, the difference integrates to zero when defining the momentum charge itself). So an agreement on the value of the energy momentum tensor cannot be reached.

Let us now summarize what we have found in this section. The FZ-multiplet, which is the minimal representation of supersymmetry containing the energy momentum tensor and supercurrent, exists in most SUSY field theories. This includes all the asymptotically free gauge theories. There are only two exceptions:

1. If some $U(1)$ gauge group has a FI-term.
2. If the geometry of the target space is nontrivial.

In these two exceptional cases, the FZ multiplet is not well-defined (in the first case due to gauge invariance and in the second case due to global issues), which means that it is impossible to embed the energy momentum tensor and supercurrent into the minimal supersymmetric multiplet.

5. Consequences

The physical input we need is the following simple claim: *If an operator is well-defined and exists in the UV of some theory then it must persist to exist in the IR.*

This allows us to derive several powerful non-renormalization theorems

1. To Be Continued