Ken Intriligator's Cargese lectures

These are some very rough lecture notes, which haven't yet been properly proof read.

1. Lecture 1

⋆ Outline. Lecture 1: intro to susy, susy breaking, sqcd, duality, susy breaking in SQCD. Lecture 2: aspects of SCFTs.

⋆ Lecture 1 references: Wess and Bagger, hep-th/9509066, hep-th/0702069.

• Susy lagrangians (4 supercharges, i.e. 4d $N=1$):

$$\begin{aligned}\mathcal{L}_{micro} = &\int d^4\theta Z_{Q_i} Q_i^\dagger e^{T_{r_i} V} Q_i \\ &+ \int d^2\theta (2\pi i \tau) \mathrm{Tr} \frac{W_\alpha W^\alpha}{32\pi^2} + \int d^2\theta W_{tree}(Q_i) + h.c..\end{aligned} \tag{1.1}$$

The Q_i are chiral superfields in representation r_i of the gauge group, and V is the vector (gauge) multiplet.

Recall chiral superfields satisfy $\bar{D}_{\dot\alpha}\Phi = 0$, so $\Phi = \phi(y) + \theta\psi(y) + \theta^2 F(y)$, where $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$. More generally, chiral operators can be defined by $[Q^\dagger_{\dot\alpha}, \Phi\} = 0$. Chiral superfields are holomorphic quantities. The Z_Q in (1.1) is the wavefunction renormalization, which is real; we don't absorb it into Q_i because that wouldn't be holomorphic.

The vector multiplet is defined by $V = V^\dagger$, with the gauge transformation law $e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda}$. The field strength is $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V = -i\lambda_\alpha(y) + D(y)\theta_\alpha - i(\sigma^\mu\bar\sigma^\nu)_\alpha^\beta\theta_\beta F_{\mu\nu}(y) + \theta^2\partial_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}$, where λ_α is the gaugino. Define the "glueball" chiral superfield $S \equiv -\mathrm{Tr} W_\alpha W^\alpha / 32\pi^2$.

• The power of homomorphy [Seiberg]. The exact superpotential is holomorphic in the fields and also the superpotential parameters (think of them as background chiral superfields). Also holomorphy in $\tau = \theta/2\pi + 4\pi i g^{-2}$. The exact W and is constrained also by the symmetries (including the broken ones, upon assigning appropriate charges to the couplings).

Example: show that the holomorphic τ has RG running at one loop only in perturbation theory. The RG running must preserve holomorphy $2\dot{\pi i \tau} = i\dot\theta + 16\pi^2 g^{-3}\beta_{g_h} = f(\tau) = -b_0$, since a constant is the only holomorphic function compatible with $\dot\theta = 0$. As we'll discuss later, β_{g_h} is related by susy to the ABJ anomaly of the $U(1)_R$ current j_R^μ which is in the same supermultiplet as $T_{\mu\nu}$. The h subscript here is to stress the distinction between the homorphic and the physical couplings. As we'll discuss later, the physical couplings include non-holomorphic effects, e.g. Z_Q, which are here hidden in the kinetic terms.

• Effective field theory: integrate out UV modes and write down theory for low-energy degrees of freedom. This is the "dual" theory. These DOF can be IR free (e.g. the pions and their chiral lagrangian nonlinear sigma model, $SU(N_f) \times SU(N_f)/SU(N_f)$) or the low-energy theory might be interacting. The latter case is a scale invariant CFT.

The effective theory is of a similar form to (1.1); for simplicity let's just write it for the case where it doesn't include gauge fields:

$$\mathcal{L}_{eff} = \int d^4\theta K(\Phi^I, \bar{\Phi}^{\bar{J}}) + W_{eff}(\Phi) + h.c., \tag{1.2}$$

where the Φ^I are some composites of the original fields, e.g. mesons and baryons. Upon integrating out the auxiliary fields, this gives a sigma model with Kähler metric $g_{I\bar{J}} = \partial^2 K/\partial\Phi^I \partial\bar{\Phi}^{\bar{J}}$: $\mathcal{L}_{scalar} = g_{I\bar{J}}\partial_\mu \Phi^I \partial^\mu \bar{\Phi}^{\bar{J}} - g^{I\bar{J}} W_I \bar{W}_{\bar{J}}$.

The theory is scale invariant if $\Delta[W] = 3$ and $\Delta[K] = 2$; in this case there must also be a conserved "superconformal" $U(1)_R$, with $R[W] = 2$ and $R[K] = 0$. Chiral superfields must have $\Delta = \frac{3}{2}R$. This will be discussed more later.

Typically, there is no way to directly derive the low-energy effective theory – but sometimes (especially using susy) one can conjecture / guess about what the IR degrees of freedom and their interactions are, and do non-trivial checks that constraints are satisfied.

For example, global symmetries, 't Hooft anomaly matching.

• Anomalies: triangle diagram with currents at vertices, only gets contributions from massless fermions (or massless scalars, with WZW terms) running in the loop (index theorems). Consider both global and gauge currents at the vertices, discuss 3 cases. Case 1: (Gauge)^3=sickness, such a theory cannot be cured unless additional matter is added to cancel the anomaly. Case 2: (Global)(Gauge)^2, this is the ABJ anomaly (since it's proportional to TrT, it's only for $U(1)$ factors), means that the global symmetry is violated by instantons; the global symmetry can still be used, as with any broken symmetry, it leads to selection rules upon assigning appropriate charge to the appropriate symmetry breaking order parameter, which in this case is Λ. Case 3: (Global)^3, this is the 't Hooft anomaly, 't Hooft argued (first at lectures here!) that it must be RG invariant (like an index).

• Gauge theories have various possible IR phases:

(a) Mass gap. Examples $\mathcal{N} = 0$ YM, $\mathcal{N} = 1$ SYM, and $\mathcal{N} = 1$ SQCD with all massive flavors. In the susy cases, we can use holomorphy etc. to write down $W_{exact}(g_p)$ to determine $\langle\Phi_p\rangle$.

(b) IR free phase. Here there are massless, IR free d.o.f., which are some composities of the UV fields. An example is ordinary QCD with N_f massless quarks, in the range where there's chiral symmetry breaking. In susy theories, there's also the phenomenon of IR free composite $SU(N_f - N_c)$ gauge fields for SQCD in the free magnetic range.

(c) CFT phase. Occurs in $SU(N_c)$ QCD with N_f massless flavors for N_f in the conformal window (whose lower boundary for N_f is being actively studied and debated by lattice gauge theorists). For SQCD the lower boundary conformal window is known from Seiberg duality: $\frac{3}{2}N_c < N_f < 3N_c$.

• A bit about susy breaking. Soft susy breaking: take Z_Q and τ in (1.1) to have θ^2 components. Gives $A_Q = Z_Q|_{\theta^2}/Z_Q$, $m_Q^2 = -Z_Q|_{\theta^4}/Z_Q$, $m_\lambda = \tau|_{\theta^2}/\tau$. This can be explicit breaking, but it's more interesting if it's spontaneous.

Global susy implies $V \geq 0$, with $V = 0$ for susy vacua and $V > 0$ for (perhaps metastable) susy breaking vacua; the vacuum energy is an order parameter for susy breaking. (One can add a negative constant, e.g. to cancel the cosmological constant, in SUGRA, since there $V_{sugra} = e^{K/M_p^2}(g^{i\bar{j}}D_iW\bar{D}_{\bar{j}}\overline{W} - 3M_p^{-2}|W|^2)$ where $D_iW = W_i + M_p^{-2}K_iW$; we'll here ignore sugra, and take $M_p \to \infty$.) *Dynamical susy breaking* if the susy breaking is generated by dimensional transmutation, can naturally get large hierarchies [Witten]: $M_s \sim \Lambda \sim M_{cutoff}e^{-c/g^2} \ll M_{cutoff}$. Since susy breaking is a property of the vacuum, it should be looked for in the low-energy effective theory, e.g (1.2).

• Our main example of susy gauge theories: SQCD, $SU(N_c)$ with N_f flavors, chiral superfields Q_i and $\widetilde{Q}_i$ in the fundamental and anti-fundamental, respectively. Give the table of fields, parameters, and their charges. Can form gauge invariant mesons $M_{f\widetilde{g}} = Q_f\widetilde{Q}_{\widetilde{g}}$. For $N_f > N_c$, can also form baryons $B \sim Q^{N_c}$, which is fully antisymmetric in the omitted flavor indices. We'll consider the theory for $W_{tree} = \mathrm{Tr}mM$, and initially consider the theory for vanishing masses m. Classically, there is then a moduli space of susy vacua, $\mathcal{M}_{cl} = \{\langle Q\rangle, \langle\widetilde{Q}\rangle | D^a = 0\}/$(gauge transformations). By a general theorem, this space is also given by $\mathcal{M}_{cl} = \{$gauge invariant chiral superfield composites$\}/$(classical relations). For $N_f < N_c$, $\mathcal{M} \cong C^{N)f^2}$, while for $N_f \geq N_c$ $\mathcal{M}_{cl}$ is a conically singular space, with $dim_C(\mathcal{M}_{cl}) = 2N_fN_c - (N_c^2 - 1)$. The classical interpretation is that the singularity is resolved upon including the massless "W-bosons" there.

Now consider the quantum theory. First consider $N_f = 0$. This theory has a mass gap and chiral symmetry breaking. Classically there is a $U(1)_R$ symmetry, which is broken by the instanton (the anomaly) to Z_{2N_c}, since $\langle S^{N_c}\rangle = \Lambda^{3N_c}$. The Z_{2N_c} chiral symmetry is then spontaneously broken to Z_2, as $\langle S \rangle = (\Lambda^{3N_c})^{1/N_c}$; these are the N_c susy vacua counted by the Witten index, $\mathrm{Tr}(-1)^F = N_c$. Can think of $\log \Lambda^{3N_c} \sim g^{-2}$ as the source coupling linearly to the operator S, and write down the 1PI effective action, with $W_{eff} = N_c(\Lambda^{3N_c})^{1/N_c}$, whose derivative w.r.t. $\log \Lambda^{3N_c - N_f}$ gives $\langle S \rangle$.

Now consider SQCD with N_f massive flavors. Initially take $m > \Lambda$. Below the scale m can integrate out the massive flavors and get SYM, so we can use the above results to see that there are N_c susy vacua with mass gap. Matching the running holomorphic coupling $g_h(\mu)$, see that $\Lambda^{3N_c - N_f} \det m = \Lambda_{low}^{3N_c}$. Then gaugino condensation generates

$$W_{low} = N_c(\Lambda_{low}^{3N_c})^{1/N_c} = (\det m \Lambda^{3N_c - N_f})^{1/N_c}. \tag{1.3}$$

This can be regarded as the 1PI effective action for the source m of the field M. Then the $m \leftrightarrow M$ Legendre transform yields

$$W_{1PI} = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{1/(N_c - N_f)}. \tag{1.4}$$

This works for any N_f for $m \neq 0$, even $N_f > N_c$ where it superficially doesn't make sense. To make sense of it there, we need to consider Seiberg duality.

Now consider $m = 0$. For the susy vacua (since the energy is zero) we can vary m/Λ without encountering any phase transitions, since it's a complex quantity we can avoid any singular points in the complex plane and there can't be any phase transition walls. For $N_f < N_c$ the superpotential (1.4) is dynamically generated, leads to a runaway vacuum. The runaway properly satisfies the condition that $V_{dyn} \to 0$ for $M \to \infty$ (asymptotic freedom ensures that the quantum effects go away when the gauge group is Higgsed far in the UV).

On the other hand, for $m = 0$ and $N_f \geq N_c$, we get $W_{dyn} = 0$, there is a quantum moduli space of susy vacua. The reason is because the symmetries require that any dynamical superpotential must depend on M as in (1.4), but the dependence on M is incompatible with the condition that $V_{dyn} \to 0$ for $M \to \infty$, unless the coefficient of W_{dyn} is zero. The classical moduli space is given by expectation values of the mesons and baryons, subject to some constraints $C(M, B, \widetilde{B}) = 0$. These spaces are singular, because we can solve $C = dC = 0$ at the origin. For $N_f = N_c$ the constraint is $C = \det M - B\widetilde{B} = 0$ and Seiberg argued that it is smoothed out by an instanton to $\det M - B\widetilde{B} = \Lambda^{2N_c}$. On the other hand for $N_f > N_c$ the symmetries don't allow any modification of the classical constraints, so $C_{quant} = C_{class}$ and $\mathcal{M}_{quant} = \mathcal{M}_{class}$. Since it's singular, there must be new d.o.f. at the origin. These new d.o.f. are given by the Seiberg dual....to be discussed next time.

2. Lecture 2

⋆ Plan: (i) finish discussing SQCD and Seiberg duality; (ii) Discuss some aspects of susy breaking, including metastable DSB in free-magnetic SQCD; (iii) Discuss SCFTs.

• Where we left off last time: SQCD with $m = 0$ and $N_f > N_c$ has a quantum moduli space of susy vacua, $\mathcal{M}_q = \mathcal{M}_{cl}$, which is singular at the origin. There must be new massless d.o.f. there. Seiberg duality: the original, "electric" theory is equivalent at low energy to the "magnetic" dual theory with gauge group $SU(N_f - N_c)$, with $W_{tree} = \Phi q\widetilde{q}$. Here $\Phi \sim M/\Lambda$. The interpretation of the duality has 2 cases

1. For $N_c < N_f < \frac{3}{2}N_c$ the dual theory is IR free. In this range the original electric theory flows in the IR to the IR free magnetic dual; that is the low-energy effective theory.

2. For $\frac{3}{2}N_c < N_f < 3N_c$, both theories are AF. They flow to the same SCFT in the IR. This is the conformal window.

Some checks: the global symmetries match, the 't Hooft anomalies match (this is a fun exercise to verify, and very convincing), the deformations and moduli spaces match (classical properties on one side map to non-perturbative quantum effects in the dual).

Discuss some aspects of susy breaking and also SCFTs.

• Return to a bit of susy breaking and dynamical susy breaking. As mentioned last time, to naturalize hierarchy problems, e.g. $m_W \ll m_{GUT}, m_{pl}$, interested in spontaneous susy breaking, where the susy breaking scale is given by $V_{min} = M_s^4$ with $M_s \sim \Lambda$. We'll now discuss metastable DSB in SQCD. Before discussing that, mention some general things and reason do expect metastable DSB.

Susy breaking and the R-symmetry problem. Nelson-Seiberg: for generic W, without an R-symmetry, susy is unbroken. For generic W with an R-symmetry, susy is broken.] The R-symmetry can be bad for phenomenology: forbids (majorana) gaugino masses. Purely spontaneous R-breaking is also bad: leads to unobserved R-axion. Consider small explicit R-breaking, with some parameter $\epsilon \ll 1$. In this case, generically get metastable susy breaking, with $\langle X \rangle_{susy} \sim 1/\epsilon^p$. The susy-breaking vacuum is now a false vacuum, but the susy breaking vacuum is far away for $\epsilon \ll 1$. The false vacuum decays by nucleating a bubble of true vacuum (with $X = X_{susy}$), like boiling, and as discussed in E. Rabinovici's lecture. Since the vacua are widely separated (and rather degenerate), the decay is parametrically suppressed for $\epsilon \ll 1$, with probability $\sim e^{-S_{bounce}} \sim e^{-1/\epsilon^p} \ll 1$.

Example: $K_{eff}(X, \bar{X})$ and $W_{eff} = \sum_{p=1}^{n} g_p X^p$. Susy is spontaneously broken if K_{eff} is regular and $n = 1$; for $n > 1$, there are $n - 1$ susy vacua. For $K = X\bar{X}$ and $W = fX$, there is a pseudomoduli space of susy breaking vacua, with ψ_X the massless goldsino. For F-term susy breaking, the goldstino is part of a chiral superfield, and its superpartner tends to be a pseudomodulus. Taking $K = X\bar{X} - c(X\bar{X})/M^2$, the goldstino superpartner is stabilized at the origin and gets a mass; the goldstino of course remains massless. There is an R-symmetry, and if $\langle X \rangle = 0$ it is not spontaneously broken, whereas if $\langle X \rangle = 0$ it is spontaneously broken and there is a massless R-axion. Adding $\Delta W = \epsilon X^2$ it is explicitly broken and there can be metastable susy breaking.

Next example, the O'R model: $W = \frac{1}{2} h X \phi_1^2 + m\phi_1\phi_2 + fX$. Note that it admits an R-symmetry, with $R[X] = R[\phi_2] = 2$, and $R[\phi_1] = 0$. Susy is spontaneously broken at tree level, with $\langle X \rangle$ a classical pseudomodulus. The pseudomodulus is lifted by $V_{CW} = \frac{1}{64\pi^2}\mathrm{Str}(\mathcal{M}^4 \log \frac{\mathcal{M}^2}{M_{cutoff}^2})$. Find that $\langle X \rangle = 0$, so the R-symmetry is not spontaneously broken. Can explicitly break the unwanted R-symmetry by adding $\Delta W = \frac{1}{2}\epsilon m \phi_2^2$, and then $\langle X \rangle_{susy} = m/h\epsilon$. Metastable susy breaking.

Next example: SQCD with $W_{tree} = mM$ in the free magnetic phase, for $m \ll \Lambda$. Analyze in the free magnetic dual, $W = \mathrm{Tr}(\Phi q\widetilde{q} - f\Phi)$, with $f = \Lambda m$. Breaks susy by the rank condition. A compact moduli space of susy breaking vacua with $\langle M \rangle = 0$ and $\langle q \rangle \neq 0$. Metastable DSB since there are the N_c susy vacua with $\langle M \rangle \neq 0$ far away (need $\epsilon = m/\Lambda \ll 1$). There is an accidental, approximate R-symmetry, which is not spontaneously broken in the susy breaking vacuum. It is violated by instantons, consistent with the above comments about explicit breaking of the R-symmetry and metastable DSB.

Note that we can't take N_f in the conformal window: the metastable susy breaking vacua wouldn't be long-lived.

• Part 2. Some phenomenology applications of CFTs or nearly CFTs:

(i) Walking technicolor. Flow near a CFT, and then away. RG walking near the CFT can help separate scales, relax problematic relations.

(ii) Suppressing flavor anarchy. Try to generate flavor hierarchies from RG running with different anomalous dimensions of different generations.

(iii) Sequestering. Try to suppress FCNC problematic Kähler potential operators via enhanced anomalous dimensions.

(iv) Unparticles. Consider phenomenology of $\mathcal{O}_{SM}\mathcal{O}_{CFT}/M^{4-\Delta_{SM}-\Delta_{CFT}}$ interaction.

(v) Helping / extending gauge mediation.

• The observables of a CFT are the operators, their dimensions, and their correlation functions.

Unitarity implies that (gauge invariant!) operator's dimensions satisfy $\Delta \geq j_1+j_2+2-\delta_{j_1j_2,0}$, where the unitarity bound is saturated for free fields. (Note that the theory is scale and conformally invariant if $T^\mu_\mu = 0$, while it can be scale but not conformally invariant if T^μ_μ is a total derivative. Unitarity using just scale symmetry gives $\Delta \geq j_1 + j_2 + 1$, but all known unitary scale invariant theories are also conformally invariant.)

• In supersymmetric theories, there is a superconformal $U(1)_R$ symmetry which is in the same supermultiplet as the stress-energy tensor. This will be discussed much more in Zohar's lecture and lecture notes. E.g. the FZ multiplet is $\mathcal{T}_{\alpha\dot{\alpha}}$, which contains the stress-energy tensor $T_{\mu\nu}$, supercurrents $S_{\alpha\mu}$ and $\bar{S}_{\dot{\alpha}\mu}$, and a global $U(1)_R$ current j^R_μ: $\mathcal{T}_\mu = j^R_\mu + \theta^\alpha S_{\alpha\mu} + \ldots + (\bar{\theta}\theta)^\nu T_{\mu\nu} + \ldots$. Conservation of the energy tensor and supercurrents is written in superspace as $\bar{D}^{\dot{\alpha}}\mathcal{T}_{\alpha\dot{\alpha}} = D_\alpha X$ where X is a chiral superfield. The F-component of X is related to the lack of scale invariance, and the lack of conservation of the $U(1)_R$ current: $\frac{2}{3}T^\mu_\mu + i\partial^\mu j^R_\mu = X|_{\theta^2}$.

Holomorphy thus links (non)conservation of the dilatation current to that of the $U(1)_R$ current. There can be non-conservation contributions both at tree-level, and from anomalies. Matching the ABJ anomaly to the imaginary part of the LHS of eqn. (2) gives the anomaly contribution $X \supset -\frac{1}{16\pi^2}\mathrm{Tr}(RG^2)\mathrm{Tr}W^2$. Using holomorphy to link the real and imaginary parts of eqns (2) and (3) gives the multiplet of anomalies $\beta(g^{-2}) = -\frac{3}{2}\frac{1}{16\pi^2}\mathrm{Tr}(RG^2)$ which has been an important puzzle, as the gauge beta function is not one-loop exact. This puzzle was discussed in many papers, with many proposed resolutions. The most fruitful and inspirational was that of Shifman and Vainshtein which ties in with the Novikov, Shifman, Vainshtein, Zakharov (NSVZ) exact beta function: using $R_i = \frac{2}{3} + \frac{1}{3}\gamma_i$ gives $\beta(g^{-2}) = -\frac{1}{32\pi^2}f_{scheme}(g^2)(3T_2(G) - \sum_i T_2(r_i)(1-\gamma_i(g^2)))$.

The holomorphic relation between the dilatation and R-current implies a corresponding relation for the charges of chiral superfields: $\Delta(Q_i) \equiv 1 + \frac{1}{2}\gamma_i(g) = \frac{3}{2}R(Q_i)$, where we'll interpret the RHS as renormalization group running R-charges when the theory is running between RG fixed points. The beta function for superpotential couplings is also related to their R-violation, as $W = h\mathcal{O}$ leads to $\beta_h = \frac{3}{2}h(R(\mathcal{O}) - 2)$. We'll now focus on the RG fixed points, where the R-symmetry is conserved.

Again, chiral superfield operators have $R(\Phi) = \frac{2}{3}\Delta(\Phi)$. So chiral, spin 0, gauge invariant operators have the unitarity bound $R \geq \frac{2}{3}$. Example: SQCD in the conformal window. The superconformal $U(1)_R$ symmetry is uniquely determined to be the anomaly free $U(1)_R$ discussed last time. Note that the gauge invariant chiral operators satisfy the unitarity bound. The meson saturates the bound at $N_f = \frac{3}{2}N_c$ showing that it is a free-field there; indeed, the entire theory is IR free magnetic there, as seen from the Seiberg dual.

• The conformal anomalies a and c. In 2d, we have $T(z)T(w) = \frac{1}{2}c(z-w)^{-4} + 2T(w)(z-w)^{-2} + \ldots$, and the central charge c (which is positive for unitary theories) is related to the conformal anomaly on a curved space, $\langle T^\mu_\mu\rangle_g = cR$, and also counts the number of d.o.f.. In 4d, the conformal anomaly is $\langle T^\mu_\mu\rangle = a(\mathrm{Euler}) + c(\mathrm{Weyl})^2$. The terms on the RHS are two combinations of the Riemann tensor squared, where the Euler term is topological (the Euler characteristic density) and the Weyl tensor vanishes in a conformally flat background. In flat space, c is related to $\langle TT\rangle$ (as in 2d), while a is related to $\langle TTT\rangle$.

(Aside: holographic theories with Einstein action have $a = c$. Adding R^2 terms, can get $a \neq c$.)

In a supersymmetric theory, a and c can be related to the superconformal $U(1)_R$ 't Hooft anomalies, as shown by Anselmi, Erlich, Freedman, Grisaru, and Johansen. This follows from the susy relation between $T_{\mu\nu}$ and j^μ_R, which implies that $X \supset \frac{c}{32\pi^2}\mathcal{W}^2 - \frac{a}{32\pi^2}\Xi$, and then holomorphy in X relates $\langle\partial_\mu j^\mu_R\rangle_A$ to $\langle T^\mu_\mu\rangle_g$, where A is a background coupling to j^μ_R, related by susy to the background metric coupled to $T_{\mu\nu}$. The result is

$$a = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R), \qquad c = \frac{1}{32}(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R). \tag{2.1}$$

Example, the SCFT obtained from SQCD in the conformal window has a and c given by these expressions, with $\mathrm{Tr}R = -N_c^2 - 1$ and $\mathrm{Tr}R^3 = N_c^2 - 1 - 2N_c^4 N_f^{-2}$.

• a-maximization. Among all possible R-symmetries, the exact superconformal R-symmetry is that which locally maximizes $a(R) = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R)$ w.r.t. R. This is proved by showing that the exact superconformal R-symmetry satisfies $9\mathrm{Tr}R^2F = \mathrm{Tr}F$ for any flavor current. (And that follows from the superspace anomaly equation $\bar{D}^2 J_F = k_{FFF}W_F^2 + k_F\mathcal{W}^2$, with no Ξ term.)

A trivial example: Consider SQCD with $R' = R + sB$, where R is the anomaly free R-symmetry with $R(Q) = R(\widetilde{Q}) = 1 - (N_c/N_f)$, and B is the baryon number symmetry, and s is a parameter. a-maximization gives $s = 0$. There are many other examples.

• Hofman-Maldacena inequalities for a/c: they consider energy flux operators, $\mathcal{E}(\widehat{n}) = \int dt r^2 n^i T^0_i(t, r\vec{n})|_{r\to\infty}$, and conjecture that their correlation functions are always non-negative. For the case of $\mathcal{N}=1$ supersymmetric theories, they find (where $\widehat{\epsilon}$ and $\widehat{n}$ are unit vectors)

$$\langle J_R\cdot\epsilon|\mathcal{E}(\widehat{n})|J_R\cdot\epsilon\rangle = 1+3\frac{c-a}{c}(\widehat{\epsilon}\cdot\widehat{n}^2-\frac{1}{3}), \tag{2.2}$$

$$\langle T\cdot\epsilon|\mathcal{E}(\widehat{n})|T\cdot\epsilon\rangle = 1+6\frac{c-a}{c}(\frac{\epsilon^*_{ij}\epsilon_{il}n_j n_l}{\epsilon^*_{ij}\epsilon_{ij}}-\frac{1}{3}), \tag{2.3}$$

where J_R is the superconformal $U(1)_R$ symmetry, T is the stress-energy tensor. The conjectured non-negativity of the RHSs then implies inequalities for the ratio a/c: for $\mathcal{N}=1$ theories, it's

$$\frac{3}{2}\geq\frac{a}{c}\geq\frac{1}{2}. \tag{2.4}$$

(similarly, for $\mathcal{N}=0$ theories it's $\frac{31}{18}\geq\frac{a}{c}\geq\frac{1}{3}$, and for $\mathcal{N}=2$ theories, it's $\frac{5}{4}\geq\frac{a}{c}\geq\frac{1}{2}$). The upper limit is always saturated by a free vector field, and the lower limit is saturated by a free matter field. The inequalities have been verified to be comfortably satisfied in every example checked.

Note from above that the energy flux is peaked in the direction $\widehat{n}$ parallel to the polarization if $c-a>0$. Most SCFTs have $c-a\geq 0$ (i.e. $\mathrm{Tr}R<0$), e.g. Banks-Zaks type theories always do.

• Current correlators. Conserved currents have

$$\langle j_{\mu I}(x)j_{\nu J}(0)\rangle = (\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\frac{C_{1,IJ}(x^2M^2)}{16\pi^4x^4}.$$

If the theory is conformal, then $C_{1,IJ}=\tau_{IJ}=-3\mathrm{Tr}RF_IF_J$ is a constant. The 1 subscript is for spin 1. The flavor currents are in supermultiplets $\mathcal{J}_I = J_I+\theta_\alpha j_{\alpha I}+\theta\bar{\sigma}^\mu\theta j_{\mu I}+\ldots$, with $D^2\mathcal{J}=\bar{D}^2\mathcal{J}=0$. The 2-point functions of the other component, $\langle JJ\rangle$, $\langle j_\alpha\bar{j}_{\dot\alpha}\rangle$, are similarly given by functions $C_{0,IJ}$ and $C_{\frac{1}{2},IJ}$. If susy is unbroken, these functions are all equal.

The two-point function of the flavor currents with the $T_{\alpha\dot\alpha}$ multiplet vanishes, so the 2-point function of flavor currents with the superconformal R-current vanishes, $\tau_{IR}=0$. This implies that the exact superconformal R-symmetry is uniquely determined by the condition that it minimizes τ_{RR} among all possible R-symmetries.

• Mention General Gauge Mediation: Buican, Meade, Seiberg, Shih relate visible sector soft masses to hidden, susy-breaking sector's current 2-point functions:

$$m_{gaugino} = \frac{g^2}{4}\int d^4x\langle Q^2(J(x)J(0))\rangle,$$

$$m^2_{sfermion} = -\frac{g^4c_2(r)}{128\pi^2}\int d^4x\log(x^2M^2)\langle Q^4(J(x)J(0))\rangle.$$

3. Supplementary material

⋆ The following is a compilation of lecture notes from previous schools. It is a supplement to this year's lectures, with additional details about various things that I won't have time to discuss much here.

4. Preliminaries

The topic of dynamical susy breaking will be introduced in lecture 1. Let's list some main relevant points.

1. $\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha} \to \langle\psi|\mathcal{H}|\psi\rangle \propto \sum_\alpha \left|Q_\alpha|\psi\rangle\right|^2 + \sum_{\dot\alpha}\left|\overline{Q}_{\dot\alpha}|\psi\rangle\right|^2 \to$ *supersymmetry is spontaneously broken iff the vacuum has non-zero energy,* $V_{vac} = M_s^4$. (Global susy only in these lectures, $M_{pl}\to\infty$. But remember that in SUGRA we can add an arbitrary negative constant to the vacuum energy, via $\Delta W = const$, so the cosmological constant can still be tuned to the observed value.)
2. Chiral superfields, $[\bar{Q}_{\dot\alpha}, \Phi\} = 0$ or in superspace $\bar{D}_\alpha\Phi = 0$, so $\Phi = \phi + \sqrt{2}\theta_\alpha\psi^\alpha + \theta^2 F +$ (derivative terms). Susy vacua can have $\langle\phi\rangle \neq 0$. If $\langle F\rangle \neq 0$, susy is broken.
3. Consider $\mathcal{L} = \int d^4\theta K(\Phi^i, \Phi^{\dagger\bar{i}}) + \int d^2\theta W(\Phi^i) + h.c.$. E.g. $K = K_{can} = \Phi^i\overline{\Phi}^{\bar{i}}\delta_{i\bar{i}}$. EOM: $\overline{D}^2\frac{\partial K}{\partial\Phi^i} + \frac{\partial W}{\partial\Phi_i} = 0$. Implies $\langle\overline{F}^{\bar{i}}\rangle = -\langle(K^{-1})^{i\bar{i}}W_i\rangle$. In components, $\mathcal{L} \supset \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + h.c. - V_F$, with $V_F = (K^{-1})^{i\bar{j}}W_i\overline{W}_{\bar{j}}$. Susy vacua must have $(K^{-1})^{i\bar{i}}W_i = 0$, for all $\bar{i}$. If inverse Kahler metric $(K^{-1})^{i\bar{i}}$ is non-degenerate (i.e. using the correct effective field theory), then this is equivalent to $\frac{\partial W}{\partial\phi^i} = 0$ for all i in susy vacua. Otherwise, susy is broken.
4. Vector superfields, $V = \ldots + \theta_\alpha\overline{\theta}_{\dot\alpha}A^{\alpha\dot\alpha} - i\bar{\theta}^2\theta_\alpha\lambda^\alpha + i\theta^2\overline{\theta}_{\dot\alpha}\overline{\lambda}^{\dot\alpha} + \frac{1}{2}\theta^2\overline{\theta}^2 D$ ($\ldots$ includes gauge d.f. and derivative terms). $W_\alpha = -\frac{1}{4}\overline{D}^2 e^{-V}D_\alpha e^V$. $W_\alpha = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2}\theta^\beta F_{\alpha\beta} + \ldots$. Glueball chiral superfield: $S = -\frac{1}{32\pi^2}\mathrm{Tr}W_\alpha W^\alpha$.
5. Classical $\mathcal{N} = 1$ susy gauge theories. $V = V_F + V_D$. $V_F = (K^{-1})^{i\bar{j}}W_i\overline{W}_{\bar{j}}$. $V_D = \frac{1}{2}\sum_a(D^a)^2$. $D_a = -g\phi^* T^a\phi$. Susy vacua must have $V_F = V_D = 0$. In addition to the F-term conditions, susy vacua have $D_a = 0$ for all $a = 1\ldots|G|$.
6. Classical $\mathcal{N} = 1$ susy gauge theories, with $W_{tree} = 0$: classical moduli spaces of vacua. $\mathcal{M}_{cl} = \{\langle\Phi\rangle|D^a = 0\}/(\text{gauge equivalence}) = \{\langle\text{gauge invt. monomials of chiral sup flds}\rangle\}/(\text{classical relations})$. The massless moduli are the chiral superfields that are left uneaten by the Higgs mechanism: $\dim_{\mathbf{C}}\mathcal{M}_{cl} = \#(\text{chiral fields}) - \#(\text{eaten})$.

7. Example: $SU(N_c)$ with $N_f = 1$ flavor, Q, $\widetilde{Q}$.

$$\mathcal{M}_{cl}: \qquad Q = \widetilde{Q}^T = (\, a \quad 0 \quad 0 \ldots 0\,).$$

Meson gauge invariant chiral superfield $M = Q\widetilde{Q} = a^2$. $\mathcal{M}_{cl} = \langle M \rangle$. Higgs mechanism: $SU(N_c) \to SU(N_c - 1)$, one chiral superfield left uneaten: $2N_c - |SU(N_c)/SU(N_c-1)| = 1$. The light field is $\sim M$. On the classical moduli space, $K_{cl} = 2\sqrt{M^\dagger M}$. Singular at origin, interpret as additional massless fields: the $SU(N_c)/SU(N_c-1)$ gauge fields.

8. $SU(N_c)$ with $N_f < N_c$. Up to gauge/flavor rotations, $\mathcal{M}_{cl}$ is given by

$$Q = \widetilde{Q} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \cdot & \\ & & & a_{N_f} \end{pmatrix}.$$

Gauge invariant description: $\mathcal{M}_{cl} = \langle M_{f\tilde{g}} \rangle$. $M_{fg} = Q_g \widetilde{Q}_{\tilde{g}}$, $f, \tilde{g} = 1 \ldots N_f$. Higgs $SU(N_c) \to SU(N_c - N_f)$. $K_{cl} \sim \sqrt{M^\dagger M}$.

9. $SU(N_c)$ with $N_f \geq N_c$. $\dim_C \mathcal{M}_{cl} = 2N_c N_f - (N_c^2 - 1)$. Up to gauge/flavor rotations,

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \cdot & \\ & & & a_{N_c} \\ & & & \\ & & & \end{pmatrix}, \quad \widetilde{Q} = \begin{pmatrix} \widetilde{a}_1 & & & \\ & \widetilde{a}_2 & & \\ & & \cdot & \\ & & & \widetilde{a}_{N_c} \\ & & & \\ & & & \end{pmatrix},$$

with $|a_i|^2 - |\widetilde{a}_i|^2 =$ independent of i. Gauge invariant description: fields $M = Q\tilde{Q}$, $B = Q^{N_c}$, $\tilde{B} = \tilde{Q}^{N_c}$, subject to classical relations. E.g. $M_{f\tilde{g}} = Q_{fc}\tilde{Q}^c_{\tilde{g}}$ (with $f, \tilde{g} = 1 \ldots N_f$ and $c = 1 \ldots N_c$) has $\mathrm{rank}(M) \leq N_c$. E.g. for $N_f = N_c$, have $\mathcal{M}_{cl} = \{M_{f\tilde{g}},\ B, \tilde{B} |\det M - B\tilde{B} = 0.\}$ Space $\mathcal{M}_{cl}$ is singular at the origin (topologically, not just its metric).

4.1. Homework

1. Consider $SO(N_c)$ with $N_f = 1$ matter field $Q \in \mathbf{N_c}$. Convince yourself that, up to $SO(N_c)$ gauge rotations, the general expectation value is $\langle Q \rangle = (a + ib, ic, 0, \ldots 0)$, with a, b, c all real. Show that the $SO(N_c)$ D terms vanish iff $ac = 0$. What is the

complex dimension of the classical moduli space? What are the independent gauge invariant operators? Verify that the dimension of the classical moduli space agrees with the Higgs mechanism counting.

2. Consider $SO(N_c)$ with N_f matter fields $Q_f \in \mathbf{N_c}$, for $f = 1 \dots N_f$. Suppose $N_f < N_c$. What are the independent gauge invariant monomials of chiral superfields? What can the gauge group be Higgsed to? Verify that the Higgs counting agrees.

5. Our main example: $\mathcal{N} = 1$ supersymmetric SQCD

As discussed in the tutorial, the gauge group is $SU(N_c)$, with matter fields $Q_f \in \mathbf{N_c}$ and $\tilde{Q}_f \in \overline{\mathbf{N_c}}$. There are equal numbers of fundamentals to satisfy the condition of no gauge anomalies $\mathrm{Tr}T^3 = 0$, where the trace is over all matter fields.

Aside: this matter content satisfies the no gauge anomaly condition by being "vector-like," meaning that all matter can be given mass terms, here via $W_{tree} = m^{f\tilde{g}} Q_f \tilde{Q}_{\tilde{g}}$. The fields Q and $\tilde{Q}$ have opposite sign $SU(N_c)$ generators, so $\mathrm{Tr}T^3 = 0$. A "chiral" theory satisfies the constraint more non-trivially, e.g. $SU(5)$ with matter $A \in \mathbf{10}$ and $\tilde{Q} \in \overline{\mathbf{5}}$ also has total gauge anomaly $\sim \mathrm{Tr}T^3 = 0$.

5.1. The symmetries

The gauge and [global] symmetries are

$$\begin{array}{ccccccc}
 & SU(N_c) & [SU(N_f)_L & SU(N_f)_R & U(1)_B & U(1)_R & U(1)_A] \\
Q & N_c & N_f & \cdot & 1 & 1-\frac{N_c}{N_f} & 1 \\
\widetilde{Q} & \overline{N}_c & \cdot & N_f & -1 & 1-\frac{N_c}{N_f} & 1 \\
M & \cdot & N_f & N_f & 0 & 2(1-\frac{N_c}{N_f}) & 2 \\
\Lambda^{3N_c-N_f} & \cdot & \cdot & \cdot & 0 & 0 & 2N_f \\
W_{eff} & \cdot & \cdot & \cdot & 0 & 2 & 0
\end{array} \tag{5.1}$$

5.2. Anomalies, instanton zero modes and charges

ABJ anomaly of global current: $\partial_\mu J^\mu = (\#)\mathrm{Tr}F\tilde{F}/32\pi^2$. Can compute # from the triangle diagram, with the global current J^μ at one vertex and the gauge fields at the other two. Can also compute # from mathematics: index of Dirac operator = number of fermion zero modes in instanton background. Each $SU(N_c)$ fundamental or anti-fundamental has # = 1 zero mode. Each $SU(N_c)$ adjoint, i.e. the gauginos, has # = $2C_2(G) = 2N_c$ zero modes. (E.g. $SU(N_c)$ SYM: the classical $U(1)_R$ is explicitly broken, by instantons, to

Z_{2N_c} (which is then spontaneously broken to Z_2 by $\langle S\rangle = \Lambda^3 e^{2\pi ik/N_c}$, $k = 1\ldots N_c$).) Each $SU(N_c)$ fundamental, e.g. each flavor of Q and $\tilde{Q}$ has 1 zero mode.

The instanton amplitude goes like $e^{-S_{inst}} = e^{-8\pi^2/g^2+i\theta}$ and the 1-loop running of the (holomorphic) gauge coupling is

$$e^{-8\pi^2/g^2(\mu)+i\theta} = \left(\frac{\Lambda}{\mu}\right)^{b_1} = \left(\frac{\Lambda}{\mu}\right)^{3N_c-N_f},$$

where b_1 is the coefficient of the 1-loop beta-function.

The $U(1)_R$ charge assignment in (5.1) is chosen to be anomaly free, which is equivalent to the fact that the instanton 't Hooft vertex, which is the vertex with the $2N_c$ gaugino zero modes λ and $2N_f$ quark zero modes, ψ_Q and $\psi_{\tilde{Q}}$ has net $U(1)_R$ charge zero (don't forget that $R(\psi_Q) = R(Q) - 1$, and the entry in the table (5.1) is $R(Q)$).

The $U(1)_A$ symmetry in (5.1) is anomalous, as the $2N_f$ quark zero modes in the instanton background has net $U(1)_A$ charge $2N_f$. Rather than thinking of $U(1)_A$ as explicitly broken by instantons, we can think of it as being spontaneously broken by assigning the instanton charge to the instanton amplitude, i.e. to $\Lambda^{3N_c-N_f}$, as in the table (5.1), and thinking of Λ as the expectation value of a background chiral superfield. Now the effective superpotential must respect $U(1)_A$ too – this is the notion of selection rules, as in the Stark effect. The effective superpotential must also be holomorphic in $\Lambda^{3N_c-N_f}$, since the dynamics doesn't know that it's not a background chiral superfield. This observation is due to Seiberg.

5.3. The exact dynamical superpotential for SQCD

Using the symmetries (5.1), we find

$$W_{dyn} \propto \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} \tag{5.2}$$

For $N_f < N_c$, this expression makes a lot of sense. Recall that the gauge group is Higgsed to $SU(N_c - N_f)$. For $N_f = N_c - 1$, the gauge group is completely Higgsed, and then there are finite action (constrained) instantons, and indeed precisely in this case (5.2) is proportional to the 1-instanton amplitude. For $N_f < N_c - 1$, (5.2) is instead associated with gaugino condensation in the unbroken $SU(N_c - N_f)$ – that is the reason for the fractional power in (5.2).

For $N_f < N_c$, the expression (5.2) moreover satisfies the boundary condition that we know from asymptotic freedom, that $W_{dyn} \to 0$ for $M/\Lambda^2 \to \infty$. However, for $N_f > N_c$ (5.2) seemingly does not satisfy this asymptotic freedom boundary condition.

For $N_f < N_c$, the classical moduli space is lifted non-perturbatively, for $N_f \geq N_c$ it is not:

$$W_{dyn} = \begin{cases} (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} & N_f < N_c \\ 0 \quad (\text{on } \mathcal{M}_{cl}) & N_f \geq N_c. \end{cases} \tag{5.3}$$

As you'll see in the exercise, there is a meaning to the analog of (5.2) for $N_f = N_c + 1$ (and higher) if it is properly interpreted. In any case, as you'll also see in the exercise the statement in (5.3) is still strictly correct.

What happens to the singularity of $\mathcal{M}_{cl}$ for $N_f \geq N_c$? Answer given by Seiberg.

For $N_f = N_c$, the symmetries allow the space $\mathcal{M}_{cl}$ to be smoothed, $\mathcal{M}_{quantum} \neq \mathcal{M}_{cl}$ (by instantons) in the quantum theory. For $N_f > N_c$, the symmetries do not allow any smoothing: $\mathcal{M}_{quantum} = \mathcal{M}_{classical}$ = singular. The singularity corresponds to new massless fields there.

For $N_f = N_c+1$, the quantum theory at the origin is given by the following *low energy effective field theory.* There are IR free fields $M_{f\tilde{g}}$, B^f, and $\tilde{N}^{\tilde{g}}$, "mesons and baryons," with *no constraints imposed.* The Kahler potential is smooth (and approximately canonical) for these fields. Evidence for this is the non-trivial 't Hooft anomaly matchings satisfied by these fields. They interact via the superpotential

$$W_{dyn} = -\frac{1}{\Lambda^{2N_c-1}}(M_{f\tilde{g}}B^f\tilde{B}^{\tilde{g}} - \det M), \tag{5.4}$$

which is irrelevant in the IR.

Exercises

1. Consider $SU(N_c)$ SQCD with $N_f < N_c$, and give all of the flavors a mass, $W_{tree} = m^{f\tilde{g}}M_{f\tilde{g}}$. The full exact superpotential is $W_{exact} = W_{dyn} + W_{tree}$, with W_{dyn} given in (5.3). Verify that there are N_c supersymmetric vacua, which are particular values of $\langle M \rangle$ which solve $\frac{\partial W_{exact}}{\partial M_{f\tilde{g}}} = 0$. Verify that these $\langle M \rangle$ can be written as $\langle M_{f\tilde{g}} \rangle = \frac{\partial W(m)}{\partial m^{f\tilde{g}}}$, with $W(m) = N_c\left(\det m\Lambda^{3N_c-N_f}\right)^{1/N_c} e^{2\pi ik/N_c}$, with $k = 1 \dots N_c$. These expressions for $\langle M \rangle$ in the N_c susy vacua are applicable also for $N_f \geq N_c$ massive flavors.

2. Consider $SU(N_c)$ with $N_f = N_c + 1$ massless flavors. The dimension of the moduli space of vacua is $\dim_C(\mathcal{M}) = 2N_fN_c - (N_c^2 - 1) = N_f^2$. As described in point 14, there are $N_f^2 + 2N_f$ massless fields at the origin (all the mesons and baryons). Verify

that the superpotential (5.4) has susy vacua which satisfy the constraints found in the previous exercise. This shows that the correct $\mathcal{M}_{cl}$ is reproduced away from the origin.

3. 't Hooft anomalies are computed from the anomaly triangle diagrams, but with global currents at each vertex. They represent obstructions to gauging global symmetries. 't Hooft argued that they must be constant along RG flows. In particular, they must match for the UV and IR fermion spectrum. Verify 't Hooft anomaly matching is satisfied, with the same 't Hooft anomalies for $SU(N_c)$ with $N_f = N_c + 1$ flavors in the UV, and the composite fields $M_{f\tilde{g}}$ and B^f and $\tilde{B}^{\tilde{f}}$ in the IR. The global symmetries (ABJ anomaly free) are $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$, and every cubic combinations of them gives a 't Hooft anomaly which must match. The charges of the IR fields are computed using $M = Q\tilde{Q}$, $B = Q^{N_c}$, and $\tilde{B} = \tilde{Q}^{N_c}$.

4. Start from $N_f = N_c + 1$ and consider giving a large mass m to one of the flavors. At low energy, we can decouple the massive flavor and recover the theory with $N_f = N_c$ flavors. The dynamical scales of the two theories are related by $\Lambda_{low}^{2N_c} = m\Lambda^{2N_c-1}$ (as found by matching the running coupling at scale m). In the effective theory (5.4), we add $W_{tree} = mM_{N_f N_f}$ to account for the mass m for the N_f-th flavor. Solve the F-term equations for M_{i,N_f}, $M_{N_f,i}$, B^{N_f}, $\tilde{B}^{N_f}$. Show that all these fields are massive (their expectation value is fixed). Show that the remaining fields have $W_{low} = (W_{dyn} + mM_{N_f N_f})|_{\langle M_{iN_f}\rangle,\ldots} = 0$, but that the remaining massless fields are constrained by $\det M - B\tilde{B} = \Lambda_{low}^{2N_c}$; this is Seiberg's quantum deformed moduli space constraint for $N_f = N_c$.

5. Start from (5.4) and add $W_{tree} = \mathrm{Tr} mM$, where m is a mass matrix for all N_f flavors. Verify that there are N_c supersymmetric vacua, given by $\langle B^f \rangle = \langle \tilde{B}_f \rangle = 0$, and $\langle M_{f\tilde{g}} \rangle$ given by the same expression found in exercise 1, simply extrapolated to $N_f = N_c + 1$.

6. Aspects of dynamical supersymmetry breaking

As we saw earlier, SQCD with $N_f < N_c$ has a dynamically generated superpotential. Classically there are supersymmetric vacua for all $\langle M \rangle$. In the quantum theory, there is no supersymmetric vacuum for any $\langle M \rangle$, except for $\langle M \rangle \to \infty$. However, this is not considered a good theory of DSB, as there is no vacuum at all for finite $\langle M \rangle$ – there is a tadpole, associated with the runaway potential $V_{dyn} \sim |W'_{dyn}|^2$. We can stop the runaway by lifting the classical moduli space, by adding masses for all of the flavors, via $W_{tree} = \mathrm{Tr} mM$, with m a $N_f \times N_f$ matrix of masses (taking all its eigenvalues to be non-zero). But as you have seen in the exercise, the full superpotential $W_{full} = W_{dyn} + W_{tree}$ now has N_c supersymmetric vacua.

We return to the question of interest for these lectures is *"How generic is dynamical supersymmetry breaking in the landscape of all possible susy gauge theories, and in the landscape of string vacua.* As will be reviewed, theories with no susy vacua seem to be very non-generic. However, theories with meta-stable DSB vacua could be much more common.

7. Dynamical SUSY breaking is non-generic

7.1. Need a Goldstino

Spontaneous SUSY breaking means there is a massless Goldstone fermion. So any candidate theory of DSB must have such a massless fermion in its spectrum. (The goldstino is eaten by the gravitino, which gets a mass, once gravity is included. We will not consider gravity effects in these lectures.)

7.2. Witten Index

All SUSY gauge theories with massive, vector-like matter have $\mathrm{Tr}(-1)_F \neq 0$ SUSY vacua. E.g. for $SU(N_c)$ SYM have $\mathrm{Tr}(-1)^F = N_c$ SUSY vacua. So for broken SUSY need a *chiral* gauge theory. (We'll review some exceptions, with massless, vector-like matter.)

7.3. Susy breaking is related to breaking global symmetries

Affleck, Dine, and Seiberg point out that a *sufficient* conditions for DSB is that

1. All non-compact flat directions are lifted (e.g. by W_{tree})
2. A global symmetry is spontaneously broken, $G \to H$.

Point 2 means there are real massless goldstone bosons, living on the compact space G/H. Point 1 ensures that they can't be promoted to complex chiral superfields, so SUSY must be broken.

7.4. DSB requires an R-symmetry, or non-generic superpotential

This was pointed out by Nelson and Seiberg. Suppose that the low-energy effective theory can be described by a supersymmetric Wess-Zumino effective Lagrangian, without gauge fields. This is the effective description, below the dynamical scale Λ, where the strong gauge dynamics binds the original microscopic fields into composites. Then DSB in the UV theory occurs if there is F-term susy breaking in this effective theory, i.e. if we can not set all $(K^{-1})^{\bar{i}i}(\partial_i W(\Phi^i) = 0$. Assuming that the Kahler metric is non-degenerate (i.e. that the low-energy effective field theory has been properly identified), this means that we can not solve all the equations

$$\frac{\partial W(\Phi^i)}{\partial \Phi^i} = 0 \qquad \text{for all } i = 1 \dots n. \tag{7.1}$$

But if W is the most generic superpotential, then (7.1) involves n equations for the n quantities Φ^i, so generally they can all be solved. Non-R flavor symmetries do not help, e.g. with a non-R global $U(1)$ symmetry, the equations (7.1) can be written as $n-1$ independent equations for $n-1$ independent unknowns, as seen by writing $W = W(\Phi^i \Phi_n^{-q_i/q_n})$, now for $i = 1 \dots n-1$. But if there is an R-symmetry, then the equations (7.1) become over-constrained: they are n equations for $n-1$ independent unknowns, as seen by writing $W = \Phi_n^{2/r_n} f(\Phi^i \Phi_n^{-r_i/r_n})$ now for $i = 1 \dots n-1$, so generically they can not be solved.

These observations fit with what we've already seen for SQCD: turning on $W_{tree} = \mathrm{Tr} mM$ breaks the R-symmetry, and indeed introduces SUSY vacua.

There can still be SUSY breaking without an R-symmetry, as the superpotential can happen to be non-generic. But it is difficult to find examples of that.

Having the R-symmetry be spontaneously broken is a sufficient condition for SUSY breaking, as in the previous subsection.

7.5. Runaway directions

Discuss runaway directions in lecture..

8. DSB is hard to analyze

Most of our techniques to analyze SUSY theories are based on holomorphy, chirality, BPS. They do not depend on the Kahler potential, which is hard to control. But finding SUSY breaking requires control of the Kahler potential. Also, since the vacuum is not supersymmetric, its dependence on the parameters might not be smooth. There can be phase transitions.

9. Examples of tree-level F-term supersymmetry breaking

Spontaneous supersymmetry breaking requires an exactly massless Goldstino fermion ψ_X. In simple models it originates from a chiral superfield X. The scalar component X can get a mass from either non-canonical Kähler potential terms, or more generally from corrections to the X propagator from loops of massive fields.

9.1. The simplest example

Consider, a theory of a single chiral superfield X, with linear superpotential with coefficient f (with units of mass^2),

$$W = fX, \tag{9.1}$$

and Kahler potential $K = K_{can} = XX^\dagger$. Supersymmetry is spontaneously broken by the expectation value of the F-component of X. The potential is $V = |f|^2$, independent of $\langle X \rangle$, so there are classical vacua for any $\langle X \rangle$. The fermion ψ_X is the exactly massless Goldstino. The complex scalar X is also classically massless. Note that there is a $U(1)_R$ symmetry, with $R(X) = 2$. For $\langle X \rangle \neq 0$ it is spontaneously broken, and the corresponding Goldstone boson is the phase of the field X.

9.2. With more general Kahler potential

Consider again (9.1), but with a more general effective Kähler potential $K(X, X^\dagger)$. The potential, $V = K^{-1}_{XX^\dagger}|f|^2$, is non-vanishing as long as the Kähler metric is non-singular. The fermion ψ_X is the exactly massless Goldstino. The vacuum degeneracy of $K = K_{can} = X^\dagger X$ is lifted by any non-trivial Kähler potential. For example, if near the origin $K = XX^\dagger - \frac{c}{|\Lambda|^2}(XX^\dagger)^2 + \ldots$, then there is a stable supersymmetric vacuum at the origin if $c > 0$. In this vacuum, the scalar component of X gets mass $m_X^2 \approx 4c|f|^2/|\Lambda|^2$. If $c < 0$, the origin is not the minimum of the potential.

The macroscopic, low-energy effective field theory must be under control to determine whether or not supersymmetry is broken. In the example (9.1), a singularity in the Kähler metric signals the need to include additional light degrees of freedom.

9.3. Additional d.o.f. can restore supersymmetry

Suppose that an additional field q becomes massless at a particular value of X, which we can take to be $X = 0$, so

$$W = hXqq + fX. \tag{9.2}$$

For $f = 0$, there is a moduli space of supersymmetric vacua, labelled by $\langle X \rangle$, and q can be integrated out away from the origin. The theory then looks similar to that of the previous subsections, except that the effective Kahler potential is singular, with $1/K_{X\bar{X}} \to 0$, at $X = 0$, corresponding to the additional massless field q there.

Turning on $f \neq 0$ lifts this moduli space. But unlike the theories of the previous subsection, the theory now no longer breaks supersymmetry, as there is a supersymmetric vacuum at $X = 0$, $q = \sqrt{-f/h}$.

Upshot: to determine whether or not supersymmetry is broken requires that the macroscopic low-energy theory be correctly identified.

Note that the theory (9.2) has a $U(1)_R$ symmetry, with $R(X) = 2$ and $R(q) = 0$. Having an R-symmetry is not a sufficient condition for SUSY breaking. The R-symmetry is not spontaneously broken, so the vacuum can be, and is, supersymmetric.

9.4. One-loop lifting of pseudo-moduli

We will be interested in the one-loop effective potential for pseudo-moduli (such as X), which comes from computing the one-loop correction to the vacuum energy

$$V_{eff}^{(1)} = \frac{1}{64\pi^2} \mathrm{STr}\, \mathcal{M}^4 \log \frac{\mathcal{M}^2}{\Lambda^2} \equiv \frac{1}{64\pi^2} \left(\mathrm{Tr}\, m_B^4 \log \frac{m_B^2}{\Lambda^2} - \mathrm{Tr}\, m_F^4 \log \frac{m_F^2}{\Lambda^2} \right), \tag{9.3}$$

where m_B^2 and m_F^2 are the tree-level boson and fermion masses, as a function of the expectation values of the pseudo-moduli.[1] In (9.3), $\mathcal{M}^2$ stands for the classical mass-squareds of the various fields of the low-energy effective theory. For completeness, we recall the standard expressions for these masses. For a general theory with n chiral superfields, Q^a, with canonical classical Kähler potential, $K_{cal} = Q_a^\dagger Q^a$, and superpotential $W(Q_a)$:

$$m_0^2 = \begin{pmatrix} W^{\dagger ac} W_{cb} & W^{\dagger abc} W_c \\ W_{abc} W^{\dagger c} & W_{ac} W^{\dagger cb} \end{pmatrix}, \qquad m_{1/2}^2 = \begin{pmatrix} W^{\dagger ac} W_{cb} & 0 \\ 0 & W_{ac} W^{\dagger cb} \end{pmatrix}, \tag{9.4}$$

with $W_c \equiv \partial W / \partial Q^c$, etc., and m_0^2 and $m_{1/2}^2$ are $2n \times 2n$ matrices.

[1] The ultraviolet cutoff Λ in (9.3) can be absorbed into the renormalization of the coupling constants appearing in the tree-level vacuum energy V_0. In particular, $\mathrm{STr}\, \mathcal{M}^4$ is independent of the pseudo-moduli.

9.5. The basic O'Raifeartaigh model

The basic model has three chiral superfields, X, ϕ_1, and ϕ_2, with classical Kähler potential $K_{cl} = X^\dagger X + \phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2$, and superpotential

$$W = \tfrac{1}{2} h X \phi_1^2 + h m \phi_1 \phi_2 - h \mu^2 X. \tag{9.5}$$

We denote the coefficient f of the linear term as $f = -h\mu^2$, with μ having dimensions of mass, to make the mass dimension explicit, and to simplify expressions. This theory has a $U(1)_R$ symmetry, with $R(X) = 2$, $R(\phi_1) = 0$, $R(\phi_2) = 2$. The tree-level potential for the scalars is, $V_{tree} = |F_X|^2 + |F_{\phi_1}|^2 + |F_{\phi_2}|^2$, with

$$F_X = h\left(\tfrac{1}{2}\phi_1^2 - \mu^2\right), \quad F_{\phi_1} = h\left(X\phi_1 + m\phi_2\right), \quad F_{\phi_2} = hm\phi_1. \tag{9.6}$$

Supersymmetry is broken because F_X and F_{ϕ_2} cannot both vanish. The X and ϕ_2 equations of motion require that $F_{\phi_1} = 0$, which fixes $\langle \phi_2 \rangle = -\langle X\phi_1/m \rangle$. The minimum of the potential is a moduli space of degenerate, non-supersymmetric vacua, with $\langle X \rangle$ arbitrary. The minimum of the potential depends on the parameter

$$y \equiv \left| \frac{\mu^2}{m^2} \right| \tag{9.7}$$

For $y \leq 1$, the potential is minimized, with value $V = |h^2 \mu^4|$, at $\phi_1 = \phi_2 = 0$ and arbitrary X. (There is a second order phase transition at $y = 1$, where this minimum splits to two minima and a saddle point.) Let us focus on the $y \leq 1$ phase.

The fermion ψ_X is the exactly massless Goldstino. The scalar component of X is a classically pseudo-modulus. The classical mass spectrum of the ϕ_1 and ϕ_2 field can be computed from (9.4). For the fermions, the eigenvalues are

$$m_{1/2}^2 = \frac{1}{4}|h|^2(|X| \pm \sqrt{|X|^2 + 4|m|^2})^2, \tag{9.8}$$

and for the real scalars the mass eigenvalues are

$$m_0^2 = |h|^2 \left(|m|^2 + \tfrac{1}{2}\eta|\mu^2| + \tfrac{1}{2}|X|^2 \pm \tfrac{1}{2}\sqrt{|\mu^4| + 2\eta|\mu^2||X|^2 + 4|m|^2|X|^2 + |X|^4} \right), \tag{9.9}$$

where $\eta = \pm 1$.

The classical flat direction of the classical pseudo-modulus X is lifted by a quantum effective potential, $V_{eff}(X)$. The one-loop effective potential can be computed from the expression (9.3) for the one-loop vacuum energy, using the classical masses (9.8) and (9.9). The pseudo-modulus X is here treated as a background. It is found that the resulting effective potential is minimized at $\langle X \rangle = 0$, so we'll simplify the expressions by just expanding around this minimum: $V_{eff} = V_0 + m_X^2 |X|^2 + \ldots$. The one loop corrected vacuum energy is

$$V_0 = |h^2 \mu^4| \left[1 + \frac{|h^2|}{64\pi^2} \left(y^{-2}(1+y)^2 \log(1+y) + y^{-2}(1-y)^2 \log(1-y) + 2 \log \frac{|hm|^2}{\Lambda^2} \right) \right]. \tag{9.10}$$

The dependence on the cutoff Λ can be absorbed into the running h. The one-loop quantum mass of the classical pseudo-modulus X is given by

$$m_X^2 = + \frac{|h^4 \mu^2|}{32\pi^2} y^{-1} \left(-2 + y^{-1}(1+y)^2 \log(1+y) - y^{-1}(1-y)^2 \log(1-y) \right). \tag{9.11}$$

The mass (9.11) indeed satisfies $m_X^2 > 0$, consistent with the minimum of the one-loop potential (9.3) being at the origin. For small supersymmetry breaking, $y \to 0$, we have

$$m_X^2 \to \frac{|h^4 \mu^4|}{48\pi^2 |m|^2}, \qquad \text{for} \qquad |\mu^2| \ll |m^2|. \tag{9.12}$$

In the limit, $y \to 1$, where the supersymmetry breaking is large, we have

$$m_X^2 = \frac{|h^4 \mu^2|}{16\pi^2} (\log 4 - 1) \qquad \text{for} \qquad |\mu^2| = |m|^2. \tag{9.13}$$

When the supersymmetry breaking mass splittings are small, the effective potential can alternatively be computed in the supersymmetric low-energy effective theory where we integrate out the massive fields ϕ_1 and ϕ_2. The effective superpotential of the low-energy theory is $W_{low} = -h\mu^2 X$, and the effective Kähler potential, $K_{eff}(X, X^\dagger)$, gets a one-loop correction from integrating out the massive fields. This gives the effective potential

$$V^{(1)} = (K_{eff\ XX^\dagger})^{-1} |h^2 \mu^4|. \tag{9.14}$$

This way of computing the effective potential is valid only when the supersymmetry breaking is small, because the true effective potential generally gets significant additional contributions from terms that involve higher super-derivatives in superspace. The effective potential (9.3) gives the full answer, whether or not the supersymmetry breaking is small. In particular, (9.14) only reproduces the effective potential (9.3) to leading order in the $y \to 0$ limit. For example, (9.14) reproduces the mass (9.12) of the small supersymmetry breaking limit, but not the mass (9.13) of the large supersymmetry breaking limit.

10. Dynamical SUSY Breaking

10.1. 3-2 model

The gauge group is $SU(3) \times SU(2)$ and we have chiral superfields: Q in $(\mathbf{3}, \mathbf{2})$, $\tilde{u}$ in $(\bar{\mathbf{3}}, \mathbf{1})$, $\tilde{d}$ in $(\bar{\mathbf{3}}, \mathbf{1})$, L in $(\mathbf{1}, \mathbf{2})$. For $W_{tree} = 0$, the classical moduli space is given by arbitrary expectation values of the gauge invariants

$$X_1 = Q\tilde{d}L \qquad , \qquad X_2 = Q\tilde{u}L \qquad , \qquad Z = QQ\tilde{u}\tilde{d}.$$

We add to the model a tree level superpotential

$$W_{tree} = \lambda Q\tilde{d}L = \lambda X_1. \tag{10.1}$$

The $SU(3)$ dynamics generates

$$W_{dyn} = \frac{\Lambda_3^7}{Z}.$$

The full superpotential is $W = W_{dyn} + W_{tree}$. This theory dynamically breaks supersymmetry. For $\lambda \ll 1$, the vacuum is at large expectation value for the fields, $v \sim \Lambda_3/\lambda^{1/7}$, where the gauge group is very much Higgsed. In this limit, we have $K \approx K_{classical}$, so the Kahler potential is under control. The vacuum energy density at the minimum is $V = M_S^4 = 3.59\lambda^{10/7}\Lambda_3^4$.

10.2. Modified moduli space example

Consider the $SU(N_c)$ theory with $N_f = N_c$ and add fields $S_{\tilde{a}}^a$, b and $\tilde{b}$ and a superpotential

$$W_{tree} = S_{\tilde{a}}^a \tilde{Q}_i^{\tilde{a}} Q_a^i + b \det \tilde{Q} + \tilde{b} \det Q$$

Classically $Q = \tilde{Q} = 0$. In the quantum theory we get the effective superpotential

$$W_{effective} = S_{\tilde{a}}^a M_a^{\tilde{a}} + b\tilde{B} + \tilde{b}B + X(\det M - B\tilde{B} - \Lambda^{2N_c})$$

which breaks SUSY.

Let's consider this for the case $N_f = N_c = 2$, where the fundamentals and anti-fundamentals can be written as $2N_f = 4$ fundamentals Q_{fc}, $f = 1 \ldots 4$, $c = 1, 2$. The gauge invariants are $M_{fg} = Q_{fc}Q_{gc}\epsilon^{cd}$, in the $\mathbf{6}$ of the global $SU(4) \cong SO(6)$ flavor symmetry. Let us write it as $\vec{M}$, to show it is in the vector of $SO(6)$. Seiberg's quantum moduli space constraint for this case is

$$\vec{M} \cdot \vec{M} = \Lambda^4. \tag{10.2}$$

We add singlets $\vec{S}$, also in the **6** of the global flavor $SO(6)$, with superpotential

$$W_{tree} = \lambda \vec{S} \cdot \vec{M}. \tag{10.3}$$

The $\vec{S}$ e.o.m. requires $\vec{M} = 0$, but that is incompatible with (10.2), so susy is broken. Note that there is a $U(1)_R$ symmetry, with $R(M) = 0$ and $R(S) = 2$.

Note that these theories provide examples of non-chiral theories that dynamically break supersymmetry. How is that compatible with the Witten index? It's because the fields $\vec{S}$ are massless. If we add to (10.3) a term $\Delta W = \frac{1}{2}\epsilon \vec{S}^2$, we find the expected $\mathrm{Tr}(-1)^F = 2$ supersymmetric, vacua at $\vec{S}^2 = \lambda^2\Lambda^4/\epsilon^2$. As we take $\epsilon \to 0$, these susy vacua run off to infinity.

At the classical level, this theory has a pseudo-moduli space of flat directions, with susy broken. To see that, note that the constraint (10.2) implies that $SO(6) \to SO(5)$, and write a solution as $\vec{M} = (\sqrt{\Lambda^4 - \vec{v}^2}, \vec{v})$, where $\vec{v}$ is an $SO(5)$ vector. Similarly, write $\vec{S} \equiv (S_1, \vec{s})$, where $\vec{s}$ is an $SO(5)$ vector. Then (10.3) is

$$W = \lambda S_1 \sqrt{\Lambda^4 - \vec{v}^2} + \lambda \vec{v} \cdot \vec{s}. \tag{10.4}$$

The vacua have $\langle S_1 \rangle$ arbitrary, and $\vec{v} = \vec{s} = 0$, with SUSY broken by $F_{S_1} \neq 0$. There is a pseudo-flat direction labeled by $\langle S_1 \rangle$. This is the Goldstino superfield, whose fermionic component is the exactly massless goldstino. The apparent $\langle S_1 \rangle$ pseudomoduli space is lifted in the quantum theory by (9.3), and the susy breaking vacuum is at $\vec{S} = 0$. The complex scalar pseudo-modulus in S_1 gets a positive mass-squared there. Note that the $U(1)_R$ symmetry is not spontaneously broken in the susy breaking vacuum, so there is no massless Goldstone boson.

10.3. An example where susy breaking is an open question

Here is an example that illustrates the need to have the effective theory under control. Consider $SU(2)$ gauge theory with a single matter field Q in the 4 dimensional representation of $SU(2)$ ($j = 3/2$). There is a 1-complex dimensional moduli space of vacua, labeled by the gauge invariant $X = Q^4$. This moduli space is lifted by the tree-level superpotential

$$W = \lambda X. \tag{10.5}$$

There is an anomaly free $U(1)_R$, with $R(Q) = 3/5$. (The $SU(2)$ instanton has 10 fermion zero modes for the field Q, as seen by noting that $\mathrm{Tr}_{j=3/2}T_z^2 = 10\mathrm{Tr}_{j=1/2}T_z^2$). There is a very non-trivial matching of $\mathrm{Tr}R$ and $\mathrm{Tr}R^3$ 't Hooft anomalies, between the microscopic $SU(2)$ and Q fields, and the macroscopic field X. This suggests that the effective field theory in the IR is described by the single composite field X as an IR free field. If so, the low-energy theory near the origin has Kahler potential

$$K_{low} = \frac{\alpha}{|\Lambda|^6} X^\dagger X + \ldots \qquad \text{near } X = 0 \tag{10.6}$$

where α is a dimensionless number that we cannot determine, the powers of Λ are on dimensional grounds, and the $\ldots$ are higher order terms, powers of $X^\dagger X$ (far from the origin, we must recover $K \approx K_{cl} \sim (X^\dagger X)^{1/4}$). Then (10.5) dynamically breaks supersymmetry, with $M_S^4 \sim |\lambda^2 \Lambda^6|$.

Note that (10.5) does not preserve the anomaly free $U(1)_R$ symmetry, but there is an accidental $U(1)_R$ symmetry of the IR free low-energy theory, with $R_{accidental}(X) = 0$, which is preserved.

However, the 't Hooft anomaly matching, suggesting an IR free spectrum and (10.6), can be a fluke. There are some known examples of misleading 't Hooft anomaly matching. The theory at the origin might be an interacting SCFT, in which case (10.6) is incorrect. In that case, the superpotential (10.5) is an irrelevant perturbation, and flows to zero in the IR, and susy is certainly unbroken.

It is not yet know which of these two scenarios is correct for this theory.

11. More homework problems

1 Verify, for the case $y = 1$, that the mass matrices have the eigenvalues (9.8) and (9.9). If you have access to mathematica, use these in (9.3) to verify (9.13).

2 Show that the superpotential (10.1) preserves a $U(1)_R$ symmetry, which is anomaly free w.r.t. both gauge groups.

3 Show that (10.1) lifts all classical flat directions.

4 Verify the TrR and TrR^3 't Hooft anomaly matching mentioned before (10.6).