Precision Counting of Black Hole Microstates

Collaborators:

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Introduction

Black holes are classical solutions in general theory of relativity.

We shall focus on time independent solutions.

A black hole in 3 space dimensions has a two dimensional compact surface – known as the event horizon – which allows objects to pass from outside to inside but not vice versa.

The event horizon has topology of S^2.

Thus objects can fall into the black hole, but nothing can ever come out of a black hole.

This notion of black holes generalizes to higher dimensional theories of gravity.

A ‘black solution’ in d space dimensions has a $(d-1)$ dimensional event horizon that allows objects to pass from outside to inside but not vice versa.

In higher dimensions ‘black solutions’ can have horizons of different topologies.

e.g. in 4 space dimensions we can have:

black holes of S^3 topology event horizon

black rings of $S^2 \times S^1$ topology event horizon.

For brevity of notation we shall refer to all of them as black holes.

Classically all black holes behave like a perfect black body at zero temperature.

Once quantum effects are taken into account, the black hole starts behaving as a black body at finite temperature.

1) Black holes emit thermal radiation at temperature:

$$T = \frac{\kappa}{2\pi}$$

in $\hbar = 1$ units.

κ: acceleration due to gravity at the horizon of the black hole

2) Black holes carry entropy:

$$S_{BH} = \frac{1}{4G_N} A$$

A: Area of the event horizon

G_N: Newton's gravitational constant

– known as the Bekenstein-Hawking entropy

In ordinary thermodynamics, entropy has a statistical interpretation.

Entropy = ln (number of quantum states which represent the same thermodynamic object)

This is usually referred to as the statistical entropy.

Can we give a similar interpretation to the black hole entropy?

One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states.

$$S_{BH} = S_{stat}$$

Strominger, Vafa

$S_{BH} = A/4G_N$, $\quad A =$ Area of event horizon

$$S_{stat} = \ln(\text{degeneracy})$$

$= \ln$ (dimension of Hilbert space of supersymmetric states carrying given set of charges)

Originally the comparison between black hole and statistical entropy was carried out in the limit of large charges.

In this limit the curvature at the horizon is small and hence we can ignore string theoretic higher derivative corrections to the effective action in computing the black hole entropy.

On the microscopic side we can use appropriate asymptotic formula for the degeneracy of states to calculate the statistical entropy.

Given this success, it is natural to carry out our study of this correspondence to finer details.

What are the effects of higher derivative corrections in the action on the black hole entropy?

Does the agreement between black hole entropy and statistical entropy continue to hold even after taking into account the effect of these higher derivative corrections?

In order to attack this problem we need to open two fronts.

First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy.

We also need to know how to calculate the statistical entropy to greater accuracy.

– calculate corrections suppressed by inverse powers of various charges.

The first problem is solved via Wald's formula.

– an explicit formula for black hole entropy in terms of geometry near the horizon in any higher derivative theory of gravity.

For extremal black holes this can be implemented via the entropy function method which gives an algebraic method for computing the entropy for any given action. A.S.

In this talk I shall address the second problem.

Our goal:

1. Find exact formula for the degeneracy of quarter BPS states in a class of $\mathcal{N}=4$ supersymmetric string theories.

We shall find that in this special class of theories the degeneracies are related to the coefficients of Fourier expansion of modular forms of subgroups of $Sp(2,\mathbb{Z})$.

2. Compare the result with black hole entropy including higher derivative corrections.

– would require studying the degeneracy for large charges, i.e. studying the

1) asymptotic behaviour of the Fourier expansion coefficients of the modular forms,

2) systematic study of corrections to the asymptotic formula

By degeneracy we shall always mean the index that counts:

No. of bosonic supermultiplets $-$ No. of fermionic supermultiplets

The underlying assumption is that once we take into account all interactions, the bosonic and fermionic supermultiplets would combine to become non-supersymmetric states, and only the index worth of states will remain supersymmetric.

CHL models based on $\mathbb{Z}_N$ orbifolds

Choudhury, Hockney, Lykken

1. Begin with heterotic string theory on

$$T^4 \times S^1 \times \widehat{S}^1$$

   T^4: A four torus

   S^1, $\widehat{S}^1$: two circles with period 2π

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 + an order N internal symmetry preserving $\mathcal{N} = 4$ supersymmetry.

Dual description

1. Begin with type IIB string theory on

$$K3 \times S^1 \times \widetilde{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 + an appropriate order N internal symmetry of type IIB string theory on $K3$.

The resulting theory is $\mathcal{N} = 4$ supersymmetric.

Special choices of N:

$$N = 1, 2, 3, 5, 7$$

$N = 1$: heterotic string theory on T^6.

The number of U(1) gauge fields in these theories is

$$r = 2k + 8, \qquad k = \frac{24}{N+1} - 2$$

For $N = 1$ we have $k = 10$ and $r = 28$.

$N = 2 \to k = 6$, $\qquad N = 3 \to k = 4$,

$N = 5 \to k = 2$, $\qquad N = 7 \to k = 1$

Although we shall focus on these theories, the analysis may be generalized for

1. other values of N,

2. $\mathcal{N}=4$ supersymmetric asymmetric $\mathbb{Z}_N$ orbifolds of type IIA on $T^4 \times S^1 \times \widehat{S}^1$ – with all supersymmetries coming from the right-moving sector

States of this theory are characterized by (r dimensional) electric charge vector Q and magnetic charge vector P.

$$Define: \qquad L = \begin{pmatrix} I_6 & \\ & -I_{r-6} \end{pmatrix}$$

Heterotic T-duality transformation is generated by a set of matrices Ω satisfying $\Omega L \Omega^T = L$ and preserving the charge lattice.

$$Q \to \Omega Q, \qquad P \to \Omega P$$

T-duality invariants

$$P^2 = P^T L P, \qquad Q^2 = Q^T L Q, \qquad P{\cdot}Q = P^T L Q$$

The heterotic S-duality group of this theory can be found by studying the T-duality group in type IIB description. Vafa, Witten

Result: It is $\Gamma_1(N)$ subgroup of SL(2,Z), consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

$a, b, c, d \in \mathbb{Z}$, $\quad a, d = 1 \text{ mod } N$, $\quad c = 0 \text{ mod } N$

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

These theories contain $6(r-6)+2$ moduli scalar fields.

1. Set of $r\times r$ matrix valued scalar M satisfying

$$M^T = M, \qquad MLM^T = L$$

2. An axion-dilaton field

$$\tau = a + iS, \qquad S > 0$$

Far away from the black hole the vacuum is parametrized by arbitrary values of these scalar fields.

$\rightarrow$ moduli space of the theory

Our goal: To find the degeneracy $d(Q,P)$ of dyons of charge (Q,P)

For the toroidal compactification a formula for $d(Q,P)$ was proposed by Dijkgraaf, Verlinde, Verlinde.

Additional arguments for this formula were provided by Shih, Strominger, Yin and by Gaiotto.

Our goal will be to

• generalize the proposal to $\mathbb{Z}_N$ CHL string theories

• prove the proposal.

We shall describe the computation for a specific class of (Q,P).

Description in IIB on $K3 \times S^1 \times \widetilde{S}^1/\mathbb{Z}_N$:

1) Q_5 D5-brane wrapped on $K3 \times S^1$

2) Q_1 D1-branes wrapped on S^1

3) $-k/N$ units of momentum along S^1

4) J units of momentum along $\widetilde{S}^1$

5) One Kaluza-Klein monopole along $\widetilde{S}^1$

– BMPV black hole at the center of Taub-NUT

After translated to the heterotic description, this gives

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2k/N, \quad Q \cdot P = J$$

Once we have calculated the degeneracy for these charges, we can extend the formula to many other charge vectors using duality symmetries of the theory.

In the weakly coupled type IIB description the low energy dynamics of the system is described by three weakly interacting pieces:

1) The closed string excitations around the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

Taking the coupling $\to$ 0 limit we can make the three pieces non-interacting.

The generating function of the spectrum of BPS states is given by the product of the generating function of each of these three different systems.

Note: Individual pieces can be interacting.

e.g. D1-D5 system binds strongly to the Kaluza-Klein monopole.

We shall denote the generating function by:

$$-1/\Phi_k(\rho,\sigma,v)$$

If we define $g(m,n,p)$ through

$$-\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p\\ m\geq -1, n\geq -1/N}} (-1)^p e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,,$$

then

$$d(Q,P) = g\left(\frac{1}{2}P^2, \frac{1}{2}Q^2, Q\cdot P\right)$$

Contribution to the generating function from individual pieces

D1-D5 center of mass motion in KK monopole background:

$$- e^{-2\pi i v} (1 - e^{-2\pi i v})^{-2}$$

$$\times \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n\sigma})^4 (1 - e^{2\pi i n\sigma + 2\pi i v})^{-2} (1 - e^{2\pi i n\sigma - 2\pi i v})^{-2} \right\}$$

Red: zero-mode contribution

$\rightarrow$ comes from studying spectrum of bound states of a superparticle to the Taub-NUT space

Pope; Gauntlett, Kim, Park, Yi

Black: non-zero mode oscillators

Dynamics of KK monopole:

$$e^{-2\pi i\sigma/N}\prod_{n=1}^{\infty}\left\{(1-e^{2\pi i n\sigma/N})^{-\frac{24}{N+1}}(1-e^{2\pi i n\sigma})^{-\frac{24}{N+1}}\right\}$$

Relative motion between the D1 and D5 branes:

$$e^{-2\pi i\rho}\prod_{r=0}^{N-1}\prod_{\substack{l,b\in Z,k'\in Z+\frac{r}{N}\\ k'\geq 0,l>0}}\left\{1-\exp(2\pi i(k'\sigma+l\rho+bv))\right\}^{-\sum_{s=0}^{N-1}e^{-2\pi i l s/N}\,c^{(r,s)}(4lk'-b^2)}$$

Done by Dijkgraaf, Moore, Verlinde, Verlinde for $N=1$.

$c^{r,s}(n)$: known coefficients, given in terms of jacobi ϑ-functions and Dedekind η-functions.

Product of three pieces $= -1/\Phi_k(\rho,\sigma,v)$

Definition of $c^{(r,s)}(n)$:

$$A = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]$$

$$B = \eta(\tau)^{-6}\vartheta_1(\tau,z)^2$$

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right]$$

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A\,,$$

$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}\,A - \frac{2}{N+1}\,B\,E_N(\tau)\,,$$

$$F^{(r,rk)}(\tau,z) = \frac{8}{N(N+1)}\,A + \frac{2}{N(N+1)}\,E_N\left(\frac{\tau+k}{N}\right)\,B\,,$$

for $1 \le s \le (N-1)$, $1 \le r \le (N-1)$, $0 \le k \le (N-1)$,

$$F^{(r,s)}(\tau,z) = \sum_{b\in\mathbf{Z},n} c^{(r,s)}(4n-b^2)\,q^n\,e^{2\pi izb}$$

Φ_k transforms as a modular form of weight k of a subgroup of the modular group of genus two Riemann surfaces.

The subgroup is $Sp(2,\mathbb{Z})$ conjugate to the group of $Sp(2,\mathbb{Z})$ matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D being 2×2 integer valued matrices satisfying

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = \mathbf{1}$$

$$C = 0 \text{ mod } N, \quad \det A = 1 \text{ mod } N, \quad \det D = 1 \text{ mod } N\,.$$

$$-\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p \\ m\ge -1, n\ge -1/N}} (-1)^p \, e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,.$$

There are two ways of carrying out this expansion.

$e^{-2\pi i v}/(1-e^{-2\pi i v})^2$ can be expanded in powers of $e^{2\pi i v}$ or $e^{-2\pi i v}$.

$\to$ for fixed m, n, the sum over p can either be bounded from above or bounded from below.

Which one gives the correct $g(m,n,p)$?

Will be discussed later.

Equivalent description

$$d(Q,P) \;=\; \frac{1}{N}(-1)^{Q\cdot P+1}\int_{\mathcal{C}} d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2+\sigma Q^2+2vQ\cdot P)\right],$$

ρ, σ, v: complex parameters

$\mathcal{C}$: a three real dimensional subspace:

$0 \le Re\,\rho \le 1, \quad 0 \le Re\,\sigma \le N, \quad 0 \le Re\,v \le 1\,.$

$$Im\,\rho = M_1, \quad Im\,\sigma = M_2, \quad Im\,v = M_3,$$
$$M_1, M_2 >> |M_3| >> 0$$

Sign of M_3 is related to whether we expand $1/\Phi_k$ in positive or negative powers of $e^{2\pi i v}$.

Using explicit computation / T-duality symmetry, the analysis can be generalized to more general charge vectors which lie on the T-duality orbit of the original charge vector.

$d(Q,P)$, expressed in terms of P^2, Q^2 and $Q\cdot P$, remains the same.

Walls of marginal stability

Moduli space of $\mathcal{N}=4$ supersymmetric string theory has walls of marginal stability.

– codimension 1 supspaces on which a quarter BPS dyon becomes marginally unstable against decay into a pair of **half BPS states**.

Across these lines of marginal stability the dyon spectrum can change.

Our results are valid for weakly coupled type IIB string theory.

– corresponds to some fixed region of the moduli space.

How does the degeneracy change as we move away from this region crossing the walls of marginal stability?

Results for walls of marginal stability

For a fixed charge vector (Q, P) there are many walls corresponding to the possiblity of decay into various pairs.

$$m(Q,P) = m(Q_1,P_1) + m(Q-Q_1, P-P_1)$$

$$Q_1 \parallel P_1, \qquad (Q-Q_1) \parallel (P-P_1)$$

– codimension 1 subspaces of asymptotic moduli space.

Let $a+iS$ denote the heterotic axion-dilaton modulus parametrizing $SL(2,R)/U(1)$.

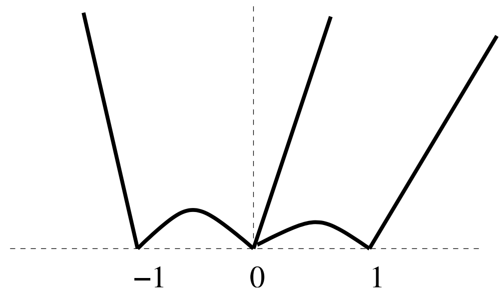
1. For fixed values of the other moduli, the walls of marginal stability are **circles** or **straight lines** in the $a+iS$ plane.

2. These curves never intersect in the interior of upper half plane.

3. They can intersect on the real axis but only at **rational points**.

$\rightarrow$ different domains bounded by walls of marginal stability have vertices at rational points or $i\infty$.

Example: $N=1$ case, i.e. heterotic on T^6



The straight line passing through 0 represents the decay $(Q,P) \to (Q,0) + (0,P)$

The straight line passing through 1 represents the decay $(Q,P) \to (Q-P,0) + (P,P)$

The circle passing through 0 and 1 represents the decay $(Q,P) \to (Q,Q) + (0,P-Q)$ etc.

Results for the degeneracy

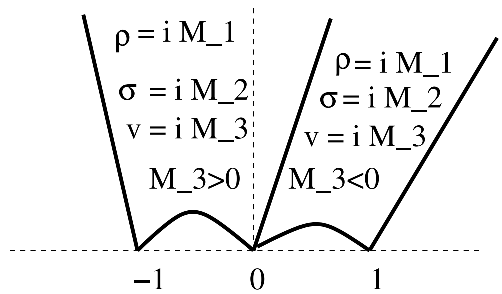
1. The degeneracies in different domains are given by the same integral formula, but with different integration contours $\mathcal{C}$.

2. As we deform one contour into another we pick up residues from the poles of $1/\Phi_k$.

– reflects change in the degeneracy across the walls of marginal stablity.

3. The change across a wall is proportional to the product of the degeneracies of a pair of half BPS dyons into which the original dyon could decay on the particular wall.

Example: $N = 1$ case, i.e. heterotic on T^6



This jump is due to a jump in the spectrum in susy quantum mechanics describing D1-D5 centre of mass motion in KK monopole background Pope; Gauntlett, Kim, Park, Yi

In other domains we have different choices of the three dimensional integration contour $\mathcal{C}$.

Comparison with black hole entropy for large charges

$$
\begin{aligned}
d(Q,P) &= \frac{1}{N}(-1)^{Q\cdot P+1}\int_{\mathcal{C}} d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \\
&\quad \exp\left[-i\pi(\rho P^2+\sigma Q^2+2vQ\cdot P)\right],
\end{aligned}
$$

a) Do the v integral by picking up residues from the poles of $1/\Phi_k$.

Result:

$$
d(Q,P)=\int d\rho d\sigma e^{-F(\rho,\sigma)}
$$

for some function $F(\rho,\sigma)$.

b) Do the ρ and σ integral using saddle point approximation.

For this we can treat

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

as we would treat a path integral and develop a Feynman diagram approach for evaluating the integral.

$\rightarrow$ 1PI effective action $-\Gamma(\rho,\sigma)$

$\ln d(Q,P)$ is the value of $\Gamma(\rho,\sigma)$ at its extremum.

$\Gamma(\rho,\sigma)$ can be calculated using Feynman diagram expansion.

Loop expansion parameter: Inverse charge

Final statistical entropy to 'one loop' order, obtained by extremizing Γ, agrees with the black hole entropy after taking into account the effect of the Gauss-Bonnet term in the effective action.

Both sides involve complicated expressions involving Dedekind η function and the match is highly non-trivial.

$$\Gamma = \frac{\pi}{2}\Big[\left(\frac{Q^2}{S}+\frac{P^2}{S}(S^2+a^2)-2\,\frac{a}{S}\,Q\cdot P\right)$$

$$+128\,\pi\,\psi_k(a,S)\Big]+\mathcal{O}(Q^{-2},P^{-2})$$

$$\rho = i/(2NS), \qquad \sigma = iN(a^2+S^2)/(2S)$$

For $\mathbb{Z}_N$ orbifolds with $N = 1, 2, 3, 5, 7$

$$\psi_k(a,S) = -\frac{1}{64\pi^2}\Big((k+2)\ln S$$

$$+\ln f^{(k)}(a+iS) + \ln f^{(k)}(-a+iS)\Big)$$

$$k = \frac{24}{N+1}-2, \qquad f^{(k)}(\tau) = \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}$$

How good is the asymptotic formula?

Q^2	P^2	$Q \cdot P$	$d(Q,P)$	S_{stat}	$S^{(0)}_{stat}$	$S^{(1)}_{stat}$
2	2	0	50064	10.82	6.28	10.62
4	4	0	32861184	17.31	12.57	16.90
6	6	0	16193130552	23.51	18.85	23.19
6	6	1	11232685725	23.14	18.59	22.88
6	6	2	4173501828	22.15	17.77	21.94
6	6	3	920577636	20.64	16.32	20.41
6	6	4	110910300	18.52	14.05	18.40

What is the role of the walls of marginal stability in this comparison?

In the large charge limit the change in the degeneracy across walls of marginal stability is exponentially suppressed compared to the leading term.

Thus we would expect that the asymptotic expansion of the black hole entropy should not change as we move across the walls of marginal stability.

Consistent with the generalized attractor mechanism.

However there is still an exponentially suppressed change in $d(Q, P)$ across walls of marginal stability.

Can we explain this on the black hole side?

It can be explained by taking into account the contribution from two centered small black holes.

Typically as we cross a wall of marginal stability, a particular 2-centered black hole (dis)-appears, causing a change in entropy.

Denef; Denef, Moore

This change is precisely in accordance with the prediction of the exact formula for $d(Q, P)$.

Speculations for dyon spectrum in $\mathcal{N}=2$ supersymmetric string theories

$$d(\vec{Q},\vec{P}) = \int_{\mathcal{C}} dM\, f(\vec{Q},\vec{P},M)$$

$$f(\vec{Q},\vec{P},M) = \exp\Big(f^{(1)}_{ij}(M) Q_i Q_j + f^{(2)}_{ij}(M) P_i P_j + f^{(3)}_{ij}(M) Q_i P_j\Big) \times g(M)$$

M: A set of complex variables

$f^{(1)}_{ij}$, $f^{(2)}_{ij}$, $f^{(3)}_{ij}$: Simple functions of M

$g(M)$: some complicated function of M that encodes non-trivial information about the spectrum

As we cross walls of marginal stability the integration contour $\mathcal{C}$ will change.

Jump in the spectrum: obtained from residues at the poles picked up during contour deformation.

Denef-Moore wall crossing formula gives

$$\Delta d(\vec{Q}, \vec{P}) = (-1)^{Q_1 \cdot P_2 - Q_2 \cdot P_1 + 1} \, |Q_1 \cdot P_2 - Q_2 \cdot P_1| \, d(\vec{Q}_1, \vec{P}_1) d(\vec{Q}_2, \vec{P}_2)$$

across a wall corresponding to the decay $(Q, P) \to (Q_1, P_1) + (Q_2, P_2)$.

Thus the wall crossing formula will relate the residues at the poles in $f(Q,P,M)$ to $f(Q_1,P_1,M)$ and $f(Q_2,P_2,M)$.

$\longrightarrow$ a set of bootstrap relations.

Can we find a consistent solution?