WALL-CROSSING FORMULA FOR BPS STATES & SOME APPLICATIONS

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1. LONG INTRO: WHAT ARE BPS STATES, AND WHY DO WE CARE ABOUT THEM?

2. SUMMARY OF RESULTS

3. SUPERGRAVITY TOOLS

4. THE ENTROPY ENIGMA

5. WALL-CROSSING FORMULA

6. APOTHEOSIS OF DONALDSON-THOMAS AND McMAHON

7. SKETCH OF A "PROOF" OF OSV

8. CONCLUSION: SOME IMPORTANT OPEN PROBLEMS.

1. INTRODUCTION

THE "SPACE OF BPS STATES" HAS BEEN A VERY USEFUL CONCEPT, CENTRAL TO MOST OF THE KEY RESULTS IN STRING THEORY + FIELD THEORY OF THE PAST 15 YEARS.

TODAY I WILL BE DESCRIBING SOME PROGRESS MADE IN COLLABORATION W/ FREDERIK DENEF OVER THE PAST 2 YEARS IN UNDERSTANDING SOME NEW ASPECTS OF BPS STATES.

SINCE THIS IS A MIXED AUDIENCE OF MATHEMATICIANS + PHYSICISTS I MUST BEGIN BY DEFINING "THE SPACE OF BPS STATES."

A. DEFINING THE "SPACE OF BPS STATES"

GENERALLY SPEAKING, SPACES OF BPS STATES ARE ASSOCIATED TO STRING/FIELD THEORIES ON ASYMPTOTICALLY AdS OR MINKOWSKIAN SPACETIME, M_n, WITH UNBROKEN EXTENDED SUPERSYMMETRY

FOR DEFINITENESS, WE FOCUS ON MINK. M_4 WITH $\mathcal{N}=2$ SUSY.

POINCARÉ SUPERALGEBRA

$$(\mathbb{C}\oplus\mathbb{R}^4\oplus\Lambda^2\mathbb{R}^4) \longrightarrow (S\otimes\mathbb{C}^2)$$

$$\hat{Z} \quad P_\mu \quad M_{\mu\nu} \qquad Q_{i\alpha}$$

$\hat{Z}$ IS A CENTRAL OPERATOR

THERE IS A HILBERT SPACE OF THE THEORY AND WE FOCUS ON THE

1-PARTICLE STATES $= \mathcal{H}$

$\mathcal{H}$ IS A REP. OF THE $d=4, \mathcal{N}=2$

AND SINCE $\hat{Z}$ IS CENTRAL WE CAN DECOMPOSE:

$$\mathcal{H} = \bigoplus_{z\in\mathbb{C}} \mathcal{H}_{\hat{Z}=z}$$

<u>LEMMA</u>: ON $\mathcal{H}_{\hat{Z}=z}$ THE HAMILTONIAN IS BOUNDED BY $E \geq |Z|$.

PROOF: $\mathcal{N}=2 \Longrightarrow$

$$\{Q_{i\alpha}, Q_{j\beta}\} = \delta_{ij}(C\Gamma^\mu)_{\alpha\beta}P_\mu + \epsilon_{ij}C_{\alpha\beta}Z$$

THIS IS A 6D SUSY ALGEBRA Q_{A},

$$\{ Q_A, Q_B \} = (C\Gamma^M)_{AB} P_M$$

WITH $P_4 + i P_5 = Z$. BUT

$$M^2 = E^2 - \vec{P}^2 - |Z|^2 \geq 0. \qquad \blacksquare$$

DEF'N: $\mathcal{H}_{BPS}$ IS THE SUBSPACE OF $\mathcal{H}$ WHERE $E = |Z|$.

<u>RMK:</u> BOUND SATURATED ONLY FOR $\vec{P} = 0$ (STATE AT REST) AND $M_{6D} = 0 \Rightarrow$ "SMALL REP'S":

$$\left(\sum c_A Q_A\right) |\psi\rangle = 0$$

AS STRESSED BY SEIBERG-WITTEN 1994 MORE RIGID, HENCE MORE COMPUTABLE.

NOW - SPECIALIZE TO TYPE II STRING THEORY ON $M_n \times X$.

- M_n IS NONCOMPACT $\Rightarrow$ TO DEFINE THE HILBERT SPACE & HAMILTONIAN WE MUST SPECIFY BOUNDARY COND'S FOR THE MASSLESS FIELDS:

$$\Phi_\infty = \lim_{\vec{x} \to \infty} \left(g_{\mu\nu}, \phi, B_{\mu\nu}, RR \right)$$

$\mathcal{H}_{\Phi_\infty}$: 1-PARTICLE HILBERT SPACE DEPENDS ON Φ_∞

- ALSO: $\mathcal{H}_{\Phi_\infty}$ IS GRADED BY SUPERSELECTION SECTORS:

$$\mathcal{H}_{\Phi_\infty} = \bigoplus_{\Gamma} \mathcal{H}^{\Gamma}_{\Phi_\infty} \qquad \Gamma \in \text{TWISTED K-THEORY}$$

REMARKS:

1. NOT TRIVIAL! NOT TRUE FOR FLUX-SECTORS!

2. ORIENTIFOLDS: TWISTED KR THEORY, DIFF. OR'S CLASSIFIED BY THE TWISTING (Distler, Freed, M.)

NOW WE PUT THESE THINGS TOGETHER:

CONSIDER IIA STRINGS ON A STATIC CY_3 X WITH FLAT B-FIELD AND FLAT RR FIELDS @ ∞.

$\Longrightarrow$ $\mathcal{N}=2$, $d=4$ SUSY

EACH $\mathcal{H}^{\Gamma}_{\Phi_\infty}$ IS A REP OF $\mathcal{N}=2$ WITH $\hat{Z} = Z(\Gamma; \Phi_\infty)$, DEFINING THE CRUCIAL CENTRAL CHARGE FUNCTION.

SO, WE STUDY THE BPS SPECTRUM

$$\mathcal{H}_{BPS} = \bigoplus_{\Gamma \in K^0(X)} \mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$$

B. DEPENDENCE ON MODULI

THE SPACES $\mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$ ARE LOCALLY CONSTANT BUT NOT GLOBALLY CONSTANT ON MODULI SPACE (= SPACE OF Φ_∞'s)

MODULI SPLIT AS:

HYPERMULTIPLETS $\times$ VECTORMULTIPLETS

[CPLX STR., ϕ, RR FIELDS] $\quad t = B + iJ$

IT HAS BEEN KNOWN FOR AT LEAST 12 YEARS (Harvey + Moore, 1995) THAT $\mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}$ CAN CHANGE AS A FUNCTION OF HYPERMULTIPLETS.

BUT THESE JUMPS OCCUR BECAUSE A VM. + HM. BECOME SIMULTANEOUSLY MASSLESS $\Longrightarrow$ THE INDEX

$$\Omega(\Gamma;\Phi_\infty) = -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}^{\Gamma}_{\Phi_\infty, BPS}} (2J_3)^2 (-1)^{2J_3}$$

IS INVARIANT.

- CODIMENSION > 1
- ASSUME GENERIC POINT IN HM MODULI SAACE AND IGNORE.

RECENT PROGRESS HAS BEEN CONCERNED WITH THE DEPENDENCE ON VECTORMULTIPLETS, IN THIS TALK,

$$t = B + iJ$$

- NOW THE INDEX Ω CHANGES AS WELL AS $\mathcal{H}$

- NOW THE JUMPING LOCUS IS REAL CODIMENSION ONE AND CANNOT BE AVOIDED.

TECHNICAL POINT:

$$\mathcal{H}^{\Gamma}_{\Phi_\infty, BPS} = \underbrace{\mathcal{H}_{\frac{1}{2}HM}}_{\text{1/2 hyper}} \otimes \mathcal{H}(\Gamma, t_\infty)$$

$2(0) + (\frac{1}{2})$ as spin rep^n

$$\Omega(\Gamma; t_\infty) = \mathrm{Tr}_{\mathcal{H}(\Gamma, t_\infty)}(-1)^F$$

HENCEFORTH FOCUS ON $\mathcal{H}(\Gamma; t_\infty)$

C. WHY DO WE CARE?

BEFORE DESCRIBING RECENT PROGRESS LET US PAUSE TO REFLECT ON MOTIVATION

PHYSICS MOTIVATION

1. THE MAIN MOTIVATION FOR RECENT WORK IS THE PROGRAM, INITIATED BY STROMINGER-VAFA (1995) OF ACCOUNTING FOR BH ENTROPY VIA MICROSTATE COUNTING. THAT GOAL IS STILL NOT FULLY ACCOMPLISHED.

IN THE PRESENT TALK THAT MEANS WE WANT TO KNOW THE ASYMPTOTICS OF $\Omega(\Gamma;t)$ FOR "LARGE Γ."

2. BPS STATES ARE CLOSELY RELATED TO TOPOLOGICAL STRING THEORY. THE MAIN GOAL HERE IS TO FIND A PHYSICALLY WELL-MOTIVATED DEFINITION OF <u>NONPTVE</u> TOPOLOGICAL STRING THEORY.

AGAIN - THIS IS RELATED TO ASYMPT'S OF $\Omega(\Gamma;t)$

3. TRADITIONALLY, BPS STATES HAVE BEEN CRUCIAL IN ESTABLISHING DUALITY ISOMORPHISMS. THE $\Omega(\Gamma;t)$ APPEAR IN QUANTUM CORRECTIONS TO S_{EFF} (LEADING TO RELATIONS TO BORCHERDS PRODUCTS, ETC.)

MATH MOTIVATION

1. PHYSICAL STABILITY OF BPS STATES IS RELATED TO MATH. STABILITY IN THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES ON A C.Y. FOLLOWING WORK OF M. DOUGLAS, T. BRIDGELAND IS DEVELOPING A THEORY OF STABLE OBJECTS. OUR RESULTS GIVE SOME PREDICTIONS/CONSTRAINTS ON WHAT WE EXPECT SHOULD BE TRUE.

2. GENERATING FUNCTIONS FOR BPS STATES INVOLVE INTERESTING ∞-DIM'L PRODUCTS – SOMETIMES RELATED TO INTERESTING AUTOMORPHIC FORMS.

3. THERE ARE SEVERAL OTHER MORE SPECULATIVE APPLICATIONS

a.) BPS ALGEBRAS – VAST GENERALIZATION OF NAKAJIMA'S THEORY.

b.) ARITHMETIC CY'S

c.) CATEGORIFICATION OF KNOT INVT'S.

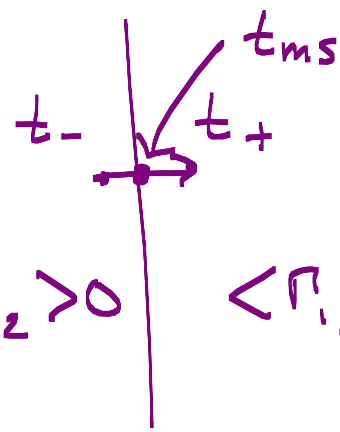
2. Summary of Results

A. WALL-CROSSING FORMULA.

$Z(\Gamma;t)$ = central charge function.

$\langle \cdot , \cdot \rangle$: symplectic product on charge space $= \int^{K} \Gamma_1 \overline{\Gamma_2}$

Def: $MS(\Gamma_1,\Gamma_2) = \{ t \mid Z(\Gamma_1,t) = \lambda Z(\Gamma_2,t) \neq 0$ and $\lambda \in \mathbb{R}_+ \}$



$\langle \Gamma_1,\Gamma_2\rangle \, \mathrm{Im}\, Z_1 \overline{Z}_2 > 0$ | $\langle \Gamma_1,\Gamma_2\rangle \, \mathrm{Im}\, Z_1 \overline{Z}_2 < 0$

Γ_1, Γ_2 PRIMITIVE $\Rightarrow$ LOSE STATES:

$$\boxed{\begin{aligned} \Delta \mathcal{H} &= (J_{12}) \otimes \mathcal{H}(\Gamma_1, t_{ms}) \otimes \mathcal{H}(\Gamma_2, t_{ms}) \\ J_{12} &= \tfrac{1}{2}\left(|\langle \Gamma_1, \Gamma_2 \rangle| - 1 \right) = \tfrac{1}{2}\left(|I_{12}| - 1 \right) \end{aligned}}$$

Appears to be a universal formula for $d=4, \mathcal{N}=2$.

Generalization: Note that

$$MS(\Gamma_1, \Gamma_2) = MS(N_1\Gamma_1, N_2\Gamma_2) \qquad N_1, N_2 \in \mathbb{Z}$$

WE CAN SAY THAT

$$\bigoplus_{N_2} u^{N_2} \Delta\mathcal{H}\Big|_{\Gamma \to \Gamma_1 + N_2\Gamma_2}$$

IS A SUPER-FOCK SPACE:

$$\mathcal{H}(\Gamma_1, t_{ms}) \bigotimes_{k} \mathcal{F}\left(u^k \left(J_{\Gamma_1, k\Gamma_2}\right) \otimes \mathcal{H}(k\Gamma_2 ; t_{ms})\right)$$

IN PARTICULAR:

$$\boxed{\begin{aligned} &\Omega_1 + \sum_{N>0} u^N \Delta\Omega(\Gamma_1 + N\Gamma_2) = \\ &= \Omega(\Gamma_1) \prod_{k>0} \left(1 - (-1)^{\langle \Gamma_1, k\Gamma_2\rangle} u^k\right)^{|\langle \Gamma_1, k\Gamma_2\rangle| \Omega(k\Gamma_2)} \end{aligned}}$$

KONTSEVICH'S TALK WILL HAVE SOMETHING TO SAY ABOUT THE CASE $N_1, N_2 > 1$.

B. AN OSV FORMULA

HERE WE HAVE BOTH A "PROOF" AND A "COUNTEREXAMPLE".

- BY A PROOF I MEAN THAT WE REDUCE THE OSV FORMULA, IN A CERTAIN CHARGE REGIME, TO TWO SEEMINGLY MORE ACCESSIBLE CONJECTURES.

- BY A COUNTEREXAMPLE I MEAN THAT WE HAVE SOME RATHER STRONG EVIDENCE THAT SUGGESTS THE OSV FORMULA IS NOT TRUE IN THE <u>WEAK</u> <u>COUPLING</u> REGIME

$$g_{top} \approx \sqrt{\frac{-\hat{q}_0}{p^3}} \lesssim \mathcal{O}(1)$$

"ENTROPY ENIGMA"

OUR VERSION OF OSV:

Let $ch\Gamma\sqrt{\hat{A}} = p^0 + P + Q + q_0$
$\in H^0 \oplus H^2 \oplus H^4 \oplus H^6$

If $ch\Gamma\sqrt{\hat{A}} = P + Q + q_0$ with P in the Kähler cone, then:

$$\boxed{\Omega(\Gamma)_\infty := \lim_{\lambda\to\infty} \Omega(\Gamma; t = B + i\lambda P)}$$

LIMIT IS WELL-DEFINED & B-INDEPENDENT.

Then

$$\Omega(\Gamma)_\infty = \int d\phi\, \mu(\phi) \left| Z^{\epsilon}_{top}(g_{top}, t)\right|^2 e^{-2\pi q \cdot \phi} \cdot \left(1 + O(e^{-\Delta})\right)$$

Where:

1. $g_{top} = \frac{2\pi}{\phi^0}$ $\quad t^A = \frac{1}{\phi^0}\left(\phi^A + i\frac{P^A}{2}\right)$

2. $Z_{top}(g,t)$ = top. string p.f.

$$= \sum_{\beta,n} N_{DT}(\beta,n)\left(-e^{-g}\right)^n e^{2\pi i \beta\cdot t}$$

3. $$Z^{\epsilon}_{top}(g,t) = \sum_{\substack{\beta\cdot P < \epsilon P^3 \\ |n| < \epsilon P^3}} (\cdot\cdot\cdot\cdot\cdot)$$

4. $$\mu(P,\phi) = \frac{1}{g^2_{top}} \mathrm{Re}\left(X^\Lambda \frac{\partial F^{\epsilon}_{top}}{\partial X^\Lambda}\right)$$

$$= \frac{1}{g^2_{top}} e^{-K} \qquad (b_1(X) = 0)$$

5. Δ = FUNCTION OF : ϵ, P, ϕ^0:

If $(g_{top})^{s.p.} \approx \sqrt{\frac{-\hat{q}_0}{P^3}} \gg 1$ $\quad \left[\hat{q}_0 = q_0 - \frac{1}{2}(D_{ABC}P^C)^{-1} Q_A Q_B\right]$

Then $$e^{-\Delta} = \exp\left(-\frac{\pi}{12\mu}\frac{\epsilon}{\phi^0}P^3\right)$$

- ABOVE ASSUMES THE TRUTH OF THE "EPS CONJECTURE" (EXPLAINED BELOW).
- MOREOVER, THE PHENOMENON OF "SWING STATES" $\Rightarrow$ WE MUST TAKE

$$\epsilon = \delta \, |P|^{-\xi_{cd}}.$$

ξ_{cd} = "Core dump exponent"
KNOWN TO BE ≤ 3.

- BUT FOR W.S. INSTANTONS TO BE RELEVANT WE NEED $\xi_{cd} \leq 2$.

$\Rightarrow$ <u>OPEN</u> <u>PROBLEM</u>.

I WILL NEXT DISCUSS THE TOOLS WE USE TO ARRIVE AT THESE RESULTS.

3. THE SUPERGRAVITY TOOLS

A: ATTRACTOR FLOW

D-BRANES ARE OBJECTS IN A CATEGORY

FOR SUSY IIA/CY – PROBABLY BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES

A BIG PROBLEM IN DISCUSSING STABILITY IN DERIVED CATEGORIES IS IDENTIFYING "STABLE OBJECTS"

WEAK COUPLING, $J \to \infty$: $\exists$ BEAUTIFUL SUGRA DESCRIPTION OF BPS STATES.

* SUPERGRAVITY ALLOWS ONE TO IDENTIFY MANY "STABLE OBJECTS" THANKS TO THE ATTRACTOR MECHANISM.

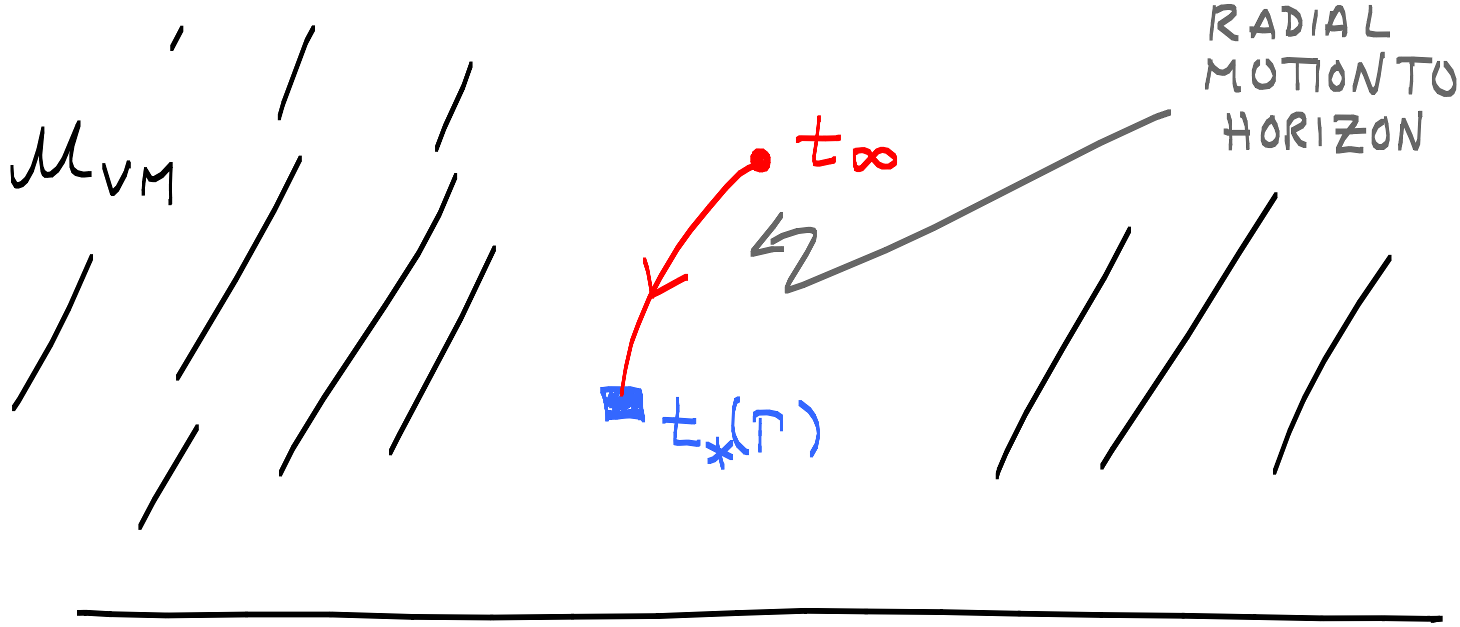
(Note: Henceforth we'll identify $\Gamma \in K(X)$ with its image $\mathrm{ch}\,\Gamma\,\sqrt{\hat{A}} \in H^{ev}(X,\mathbb{Q})$)

ATTRACTOR MECHANISM: (F.K.S. ; STROMINGER)

Γ, t_∞ & SPHERICAL SYMMETRY

$\Longrightarrow$ UNIQUE BPS SOLUTION

SCALAR FIELDS $t = t(r)$



ATTRACTOR FLOW = GRADIENT FLOW FOR

$$\log |Z(\Gamma;t)|^2$$

$$Z(\Gamma;t) := \langle \Gamma, \omega \rangle$$

$\langle \cdot , \cdot \rangle$:= SYMPLECTIC PRODUCT ON $K^0(X)$:

$$\int \mathrm{ch} E \, \mathrm{ch} \overline{E'} \, \hat{A} = \int -p^0 q_0' + PQ' - QP' + q_0 p^{0\prime}$$

$$\omega \approx e^{\frac{K}{2}} e^{t} = e^{\frac{K}{2}} e^{B+iJ}$$

$$Z = -e^{+K/2} \int e^{-t} \Gamma$$

$$K = -\log (\mathrm{Im}\, t)^3$$

$$\boxed{Z \approx \frac{\frac{1}{6}p^0 t^3 - \frac{1}{2}P t^2 + Qt - q_0}{\sqrt{(\mathrm{Im}\, t)^3}}}$$

DEF'N: $Z_*(\Gamma) = Z(\Gamma; t_*(\Gamma)) \neq 0$

$\Rightarrow t_*(\Gamma)$ IS A **REGULAR** **ATTRACTOR POINT**

HORIZON AREA $= 4\, S(\Gamma) = 4\pi |Z_*(\Gamma)|^2$

$$S(\Gamma) = \sqrt{\mathcal{D}(\Gamma)}, \quad \underbrace{\mathcal{D} : H^{ev}(X, \mathbb{R}) \to \mathbb{R}}_{\text{"DISCRIMINANT"}}$$

CONJECTURE: $\log |\Omega(\Gamma; t_*(\Gamma))| \sim S(\Gamma)$

FOR "LARGE" Γ

EXAMPLE:

$\Gamma = P + Q + q_0$ i.e. $p^0 = 0$

$$S(\Gamma) = 2\pi \sqrt{\frac{-\hat{q}_0\, \chi(P)}{6}}$$

$$\chi(P) := P^3 + c_2 \cdot P$$

$$\hat{q}_0 := q_0 - \frac{1}{2}\left(D_{ABC}P^C\right)^{-1} Q_A Q_B$$

REGULAR ATTRACTOR POINT FOR

- P IN KÄHLER CONE
- $\hat{q}_0 < 0$

DIGRESSION: ROUGH MICROSCOPIC DESCRIPTION:

SINGLE D4 WRAPS $\Sigma \in |P|$ & HAS FLUX $F \in H^2(\Sigma, \mathbb{Z})$ AND N $\overline{D0}$'s

$\chi(P) = P^3 + c_2 \cdot P$ = EULER CHARACTER OF Σ

> 0 FOR $P \in$ KÄHLER CONE

D-BRANES ARE A SOURCE OF RR CURRENT:

$$J = ch\, \mathcal{E} \sqrt{\hat{A}} \Rightarrow$$

$$\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N$$

N.B.! CAN BE POSITIVE...

$\hat{q}_0 > 0$ STATES: KNOWN AS "POLAR STATES"

$\text{SUSY} \Longleftrightarrow \quad F^{2,0} = 0$

FIXES MODULI OF Σ: "OPEN STRING VACUA"

$$\Sigma \in NL(P,F) = \{ \Sigma \in |P| \;\big|\; F \in H^{1,1} \}$$

NOTE

- $b_2(\Sigma) = \chi(P) + 2 = P^3 + c_2 \cdot P + 2 \gg b_2(X)$
- "GENERIC" $F \Rightarrow NL(P,F) =$ FINITE SET OF POINTS

$\mathcal{M}(P,F,N) =$ MODULI OF D4's

= "MODULI OF STABLE OBJECTS IN THE DERIVED CATEGORY WITH SPECIFIED CHERN CLASSES."

ROUGHLY:

$$\begin{tikzcd} \mathrm{Sym}^N \Sigma \arrow[r] \arrow[d] & \mathcal{M}(P,F,N) \arrow[d] \\ \underset{\text{smooth}}{\Sigma} \arrow[r, hook] & NL(P,F) \end{tikzcd}$$

BPS STATES : $\sum c_A Q_A |\psi\rangle = 0$

$\Longrightarrow$ $|\psi\rangle$ COHO CLASS OF $\mathcal{M}(P,F,N)$

$$d(F,N) = \chi(\mathcal{M}(P,F,N))$$

= SOME KIND OF

DONALDSON-THOMAS INVARIANT

$\Omega(\Gamma)_\infty$ = FINITE SUM OF $d(F,N)$'s

RETURN TO SUGRA PERSPECTIVE

WHAT HAPPENS IN SUGRA IF $\hat{q}_0 > 0$?

$\mathcal{D}(\Gamma) < 0$ FOR $\hat{q}_0 > 0$!!

ATTRACTOR SOLUTION DOES NOT EXIST:

$\mathcal{D}(\Gamma) < 0 \Longleftrightarrow Z(\Gamma; t)$ HAS A ZERO.

FLOW REACHES THE ZERO AT $r = r_c$:

$$\otimes \; t_0(\Gamma): \quad Z(\Gamma; t_0(\Gamma)) = 0$$

FLOW DOES NOT CONTINUE BEYOND $r_c > 0$

SO, NO BPS STATE ..... ?

INDEED, SOMETIMES NOT, BUT

DENEF: NOT NECESSARILY! RELAX SPHERICAL SYMMETRY $\Rightarrow$

B. MULTICENTERED SOLUTIONS:

GENERAL BPS EQUATIONS

(1.) $ds^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} d\vec{x}^2$

$U = U(\vec{x})$, $\vec{x} \in \mathbb{R}^3$

(2.) $2e^{U}\,\mathrm{Im}\left(e^{-i\alpha}\omega\right) = -H(\vec{x})$

$\alpha(\vec{x}) = \arg\left(Z(\Gamma; t(\vec{x}))\right)$

$H: \mathbb{R}^3 \longrightarrow H^{ev}(X, \mathbb{R})$ HARMONIC

$$H(\vec{x}) = \sum_j \frac{\Gamma_j}{|\vec{x}-\vec{x}_j|} - 2\,\mathrm{Im}\left(e^{-i\alpha}\omega\right)_{\vec{x}=\infty}$$

$$\boxed{(2) \Longrightarrow t(\vec{x}) \text{ AND } U(\vec{x}) \text{ ARE DETERMINED BY } \Gamma_j, \vec{x}_j}$$

(3.) $\quad *_3 d\,\textcircled{H} = \langle dH, H\rangle$

$\Longrightarrow$ INTEGRABILITY CONDITION:

$$\sum_{\substack{j \\ j\neq i}} \frac{\langle \Gamma_i, \Gamma_j\rangle}{|\vec{x}_i - \vec{x}_j|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_\infty$$

SUGRA SOLUTION EXISTS $\Longleftrightarrow$

$\forall\, \vec{x} \in \mathbb{R}^3$:

$t(\vec{x}) \in \mathcal{M}_{VM}$ & $\mathcal{D}\left(H(\vec{x})\right) > 0$,

i.e.

$$\pi e^{-2U(\vec{x})} = S(H(\vec{x})) \geqslant 0$$

(A VERY NONTRIVIAL CONDITION TO CHECK ...)

EXAMPLE: TWO CENTERS

$$\overset{\Gamma_1}{\bullet} \xrightarrow{\vec{R}} \overset{\Gamma_2}{\bullet}$$

$$R = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\mathrm{Im}(Z_1 \bar{Z}_2)_\infty}$$

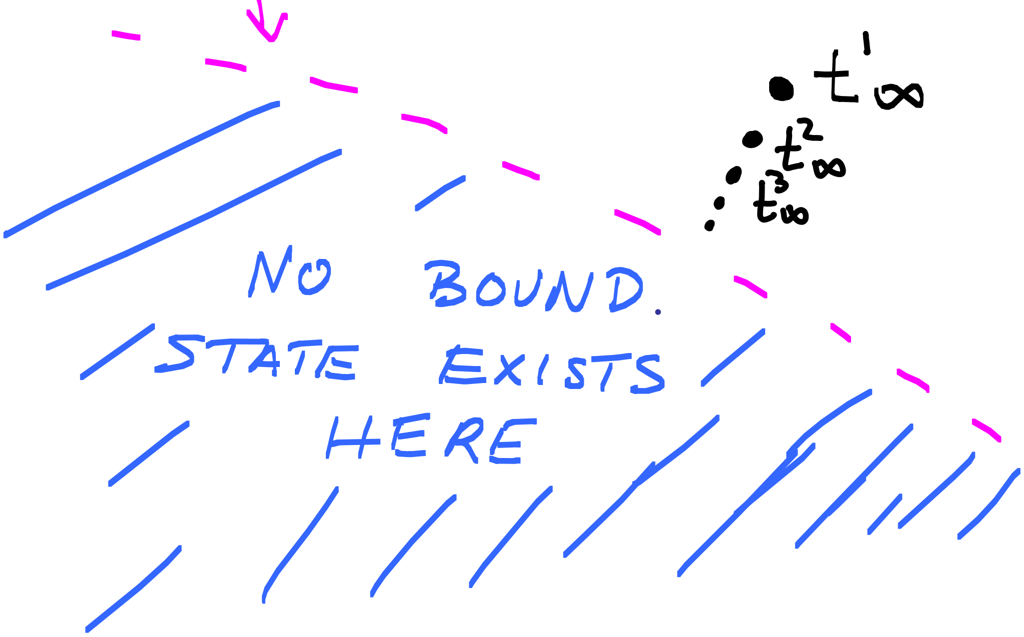
$$\vec{J} = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \hat{R} \quad \text{(USEFUL LATER)}$$

- NOTE: $\langle \Gamma_1, \Gamma_2 \rangle \mathrm{Im}(Z_1 \bar{Z}_2)_\infty > 0$

- NOTE THAT BY CHANGING t_∞ WE CAN MAKE $\mathrm{Im}(Z_1 \bar{Z}_2)\big|_{t_\infty} \to 0$ WHILE $|Z_1 + Z_2|_{t_\infty} \neq 0$

ILLUSTRATES THE KEY POINT OF MARGINAL STABILITY:

$$MS(\Gamma_1,\Gamma_2) := \left\{ t \in \mathcal{M}_{VM} \;\middle|\; \frac{Z_1}{Z_2} \in \mathbb{R}_+ \right\}$$

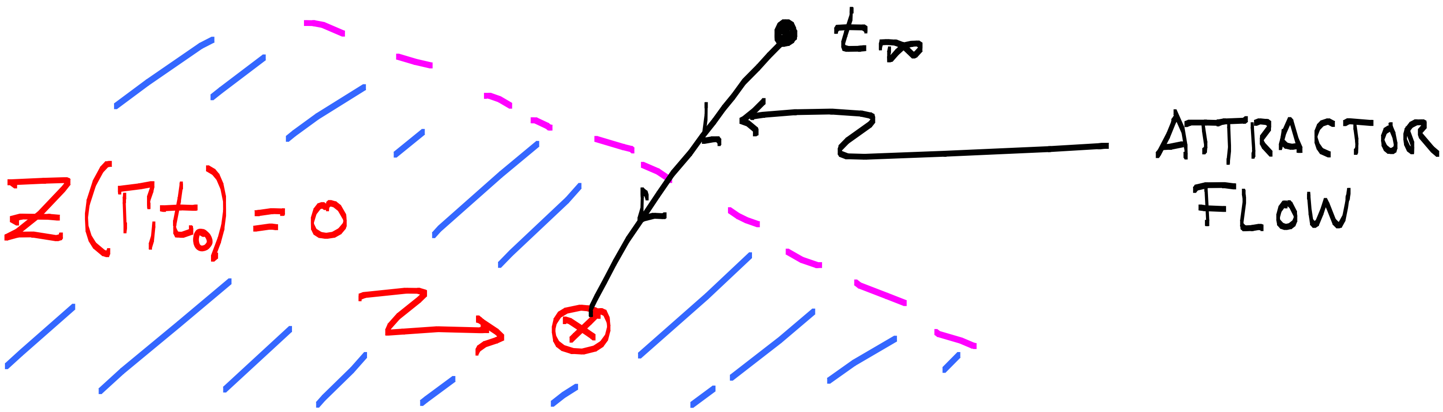


CHANGE BC's @ $r=\infty \Rightarrow$

$R_{12} \longrightarrow \infty$

RECALL:

$$\mathcal{D}(\Gamma) < 0 \iff Z(\Gamma;t) \text{ HAS A ZERO IN } \mathcal{M}_{VM}$$



EXAMPLE:

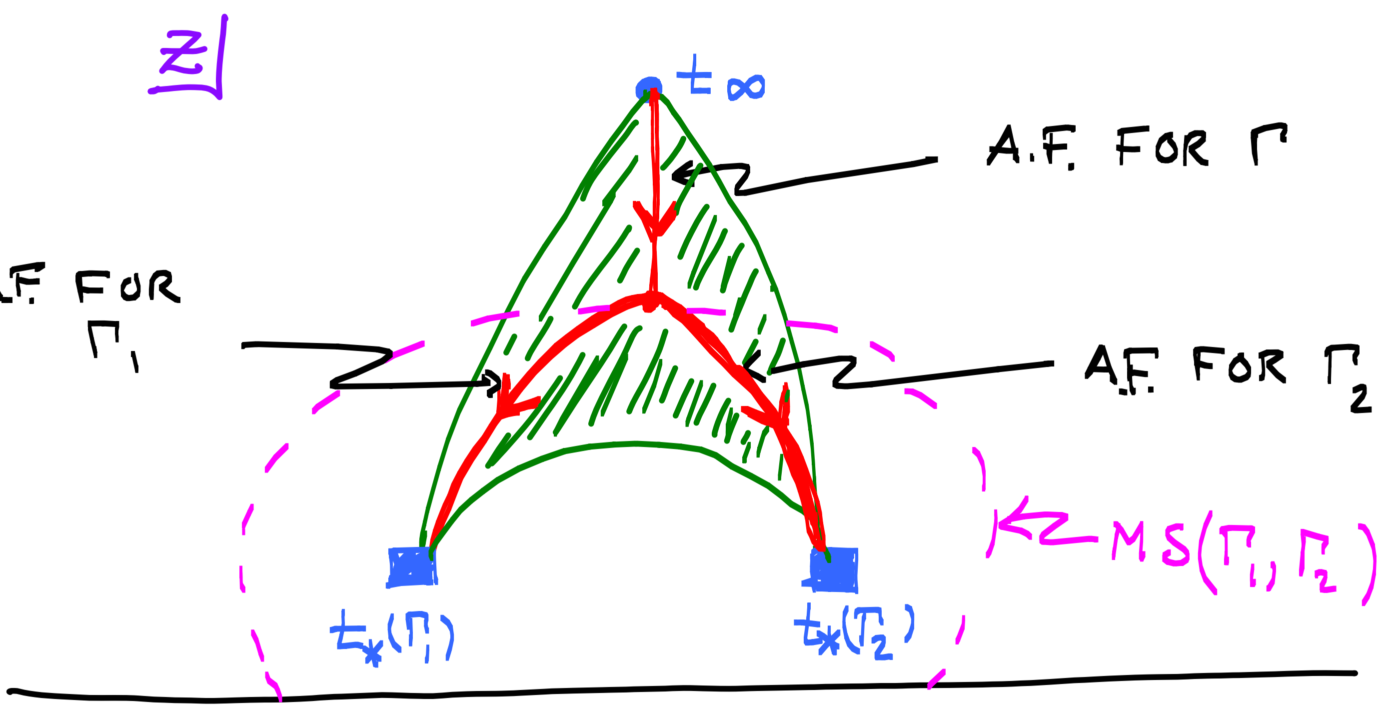
$$\Gamma = (0, P, 0, q_0) = \Gamma_1 + \Gamma_2$$

$$\Gamma_1 = (r, \tfrac{1}{2}P, Q, \tfrac{1}{2}q_0) \quad \Gamma_2 = (-r, \tfrac{1}{2}P, -Q, \tfrac{1}{2}q_0)$$

WE CAN HAVE $\mathcal{D}(\Gamma) < 0$ WHILE Γ_1, Γ_2 SUPPORT REGULAR ATTRACTOR POINTS.

$\Rightarrow$ SUSY D6 $\overline{\text{D6}}$ BOUNDSTATE

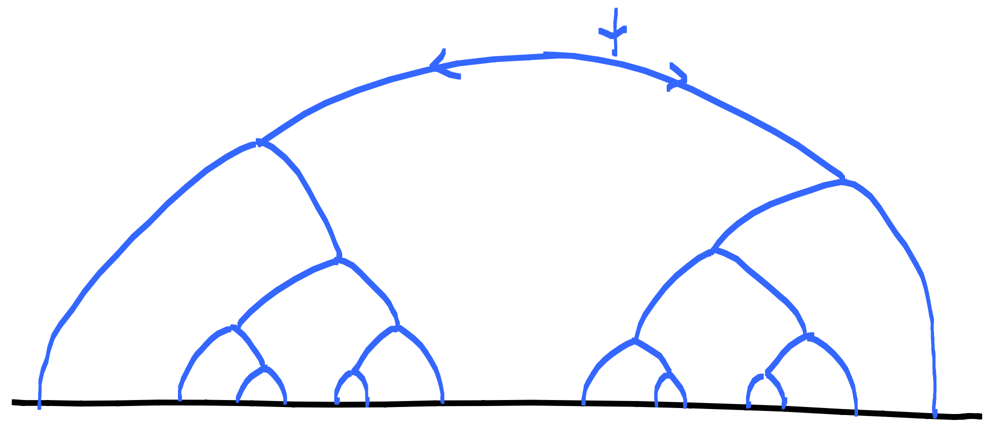
PLOT THE IMAGE OF $t(\vec{x}) \in \mathcal{M}_{VM}$ FOR A TWO-CENTERED SOL'N: $t = z\underline{P}$



C : SPLIT ATTRACTOR FLOWS

<u>DEF'N</u>: A "<u>SPLIT</u> <u>ATTRACTOR</u> <u>FLOW</u>" IS A PIECEWISE ATTRACTOR FLOW TREE CONNECTED AT WALLS OF MARGINAL STABILITY AND TERMINATNG ON <u>REGULAR</u> ATTRACTOR POINTS.

THESE S.A.F.'S CAN BE VERY INTRICATE. FOR EXAMPLE, WE CONSTRUCTED AN INFINITE FRACTAL FAMILY OF FLOWS:



SPLIT ATTRACTOR CONJECTURE (DENEF)

(a.) (COMPONENTS OF MODULI OF) MULTICENTERED SOLUTIONS ARE IN $1 \leftrightarrow 1$ CORRESPONDENCE WITH S.A.F.'S.

(b.) FOR A FIXED (t_∞, Γ) THERE ARE A FINITE NUMBER OF S.A.F.'S

- USEFUL BECAUSE CHECKING $\mathcal{D}(H(\vec{x})) > 0$ IS DIFFICULT

- $\mathcal{H}_{BPS}$ IS PARTITIONED BY SPLIT ATTRACTOR FLOWS

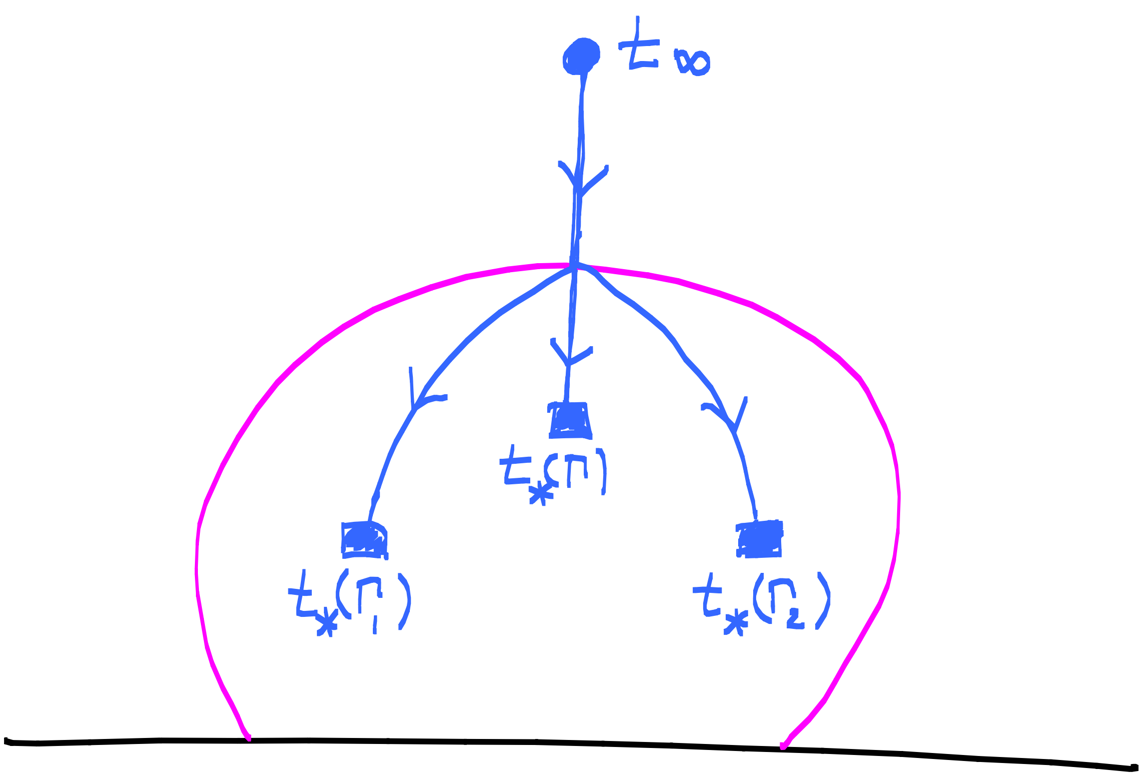
- $\exists$ SOME INTERESTING OPEN PROBLEMS HERE ....
  - QUANTUM MIXING BETWEEN DIFFERENT AF TREES
  - USEFUL EXISTENCE CRITERION FOR SCALING SOLUTIONS.

4. THE ENTROPY ENIGMA

$$\Gamma = (0, P, 0, q_0) = \Gamma_1 + \Gamma_2$$

$$\Gamma_1 = \left(r, \tfrac{1}{2}P, Q, \tfrac{1}{2}q_0\right) \quad \Gamma_2 = \left(-r, \tfrac{1}{2}P, -Q, \tfrac{1}{2}q_0\right)$$

FOR AN APPROPRIATE RANGE OF t_∞, Q, q_0 $\exists$ <u>BOTH</u> SINGLE-CENTERED <u>AND</u> TWO-CENTERED SOLUTIONS



SO... COMPARE ENTROPIES

$$S(\Gamma) \quad \text{vs.} \quad S(\Gamma_1) + S(\Gamma_2)$$

IN FACT,

$\exists$ FAMILY OF CHARGES

$$\lambda\Gamma = \lambda(0, P, 0, q_0) = \Gamma_1^\lambda + \Gamma_2^\lambda$$

$$\Gamma_1^\lambda = \left(r, \tfrac{\lambda}{2}P, \lambda^2 Q, \tfrac{\lambda}{2}q_0\right) \quad \Gamma_2^\lambda = \left(-r, \tfrac{\lambda}{2}P, -\lambda^2 Q, \tfrac{\lambda}{2}q_0\right)$$

$S(\lambda\Gamma) = \lambda^2 S(\Gamma)$ BUT

$$\boxed{S(\Gamma_1^\lambda) = S(\Gamma_2^\lambda) \sim \frac{(\lambda P)^3}{r} \sim \lambda^3}$$

SOME TECHNICAL DETAILS

1. CONSTRUCT A FAMILY OF 2-CENTERED

$$\tilde{\Gamma}_1^{\lambda} = \left(r, \frac{P}{2}, Q, \lambda^{-2}\frac{q_0}{2}\right)$$

$$\tilde{\Gamma}_2^{\lambda} = \left(-r, \frac{P}{2}, -Q, \lambda^{-2}\frac{q_0}{2}\right)$$

$\tilde{\Gamma}_i^{\lambda}$ CAN BE 1-CENTERED BH'S OR CAN THEMSELVES BE POLAR

2. ATTRACTOR FORMALISM HAS A SCALING SYMMETRY UNDER

$$T_\lambda (p^0, P, Q, q_0) = (p^0, \lambda P, \lambda^2 Q, \lambda^3 q_0)$$

$$S(T_\lambda \Gamma) = \lambda^3 S(\Gamma)$$

3. APPLY TO $\quad T_\lambda \tilde{\Gamma}_1^{\lambda} + T_\lambda \tilde{\Gamma}_2^{\lambda} = \lambda \Gamma$

DEGENERACY DICHOTOMY

- WE HAVE FOUND CONTRIBUTIONS TO $\Omega(\lambda\Gamma)_\infty$ GROWING LIKE e^{λ^3}

- IF INDEED $\Omega(\lambda\Gamma)_\infty \sim e^{\lambda^3}$ THEN WEAK COUPLING OSV IS WRONG.

- BUT $\Omega(\lambda\Gamma)_\infty$ IS AN INDEX. IT IS POSSIBLE THAT

$$\Omega(\lambda\Gamma)_\infty = \sum \pm e^{\lambda^3} \sim e^{\lambda^2}$$

- THIS RAISES THE QUESTION OF $\dim \mathcal{H}(\Gamma;t)$ vs. $\Omega(\Gamma;t)$

  * Physically the dimension is relevant
  * All tests use the index
  * We have ignored nonperturbative string effects which could lift e^{λ^3} states leaving e^{λ^2} ground states.

$E-|Z| = 0 \qquad \sim e^{\lambda^2}$ states

$\Rightarrow$ $E-|Z| \sim e^{-1/g_s} \qquad \sim e^{\lambda^3}$ states

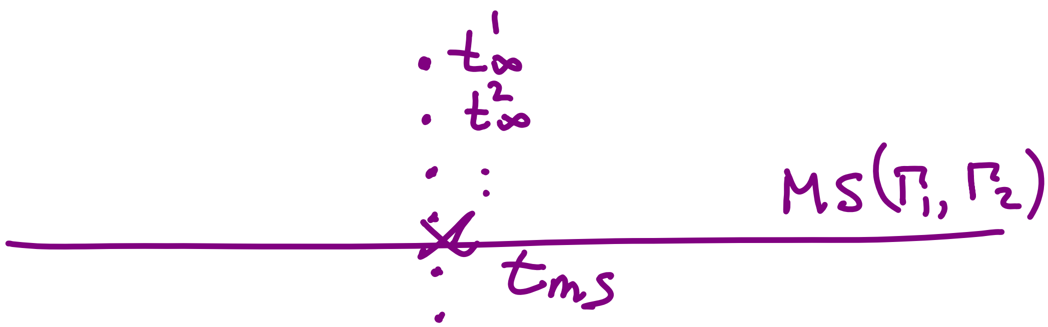
5. WALL-CROSSING FORMULA

RETURN TO 2-CENTER BOUNDSTATE

$$R_{12} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{|Z_1 + Z_2|_\infty}{\mathrm{Im}(Z_1 \bar{Z}_2)_\infty}$$

$$J_{12} = \frac{1}{2}\left(|\langle \Gamma_1, \Gamma_2 \rangle| - 1 \right)$$

quantum correction



Γ_1, Γ_2 PRIMITIVE $\Rightarrow$ STATES LOST FROM $\mathcal{H}(\Gamma; t_\infty)$ ARE

$$(J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{MS}) \otimes \mathcal{H}(\Gamma_2; t_{MS})$$

ARGUMENT FOR UNIVERSALITY

1. ON $MS(\Gamma_1, \Gamma_2)$ CAN ADIABATICALLY SEPARATE CONSTITUENTS.

2. ONLY LONG RANGE FIELDS CAN ADD EXTRA D.O.F.

3. CLASSICAL FORMULA FOR $\vec{J}$ CARRIED BY EM FIELD OF A PAIR OF DYONS

$$\vec{J} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \hat{r}$$

4. QUANTUM CORRECTION IS FIXED BY $\Delta \mathcal{H} = 0$ WHEN $\langle \Gamma_1, \Gamma_2 \rangle = 0$.

APPLICATIONS

A. COMPUTATION OF Ω'S:

$$\Delta\Omega = (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\,|\langle\Gamma_1,\Gamma_2\rangle|\,\Omega(\Gamma_1)_{t_{ms}}\,\Omega(\Gamma_2)_{t_{ms}}$$

$$\Omega(\Gamma;t_\infty) = \sum_{\substack{\text{TREES}\\ T}} \prod_{\Gamma_a\to\Gamma_b+\Gamma_c} (-1)^{\langle\Gamma_b,\Gamma_c\rangle-1}\,|\langle\Gamma_b,\Gamma_c\rangle| \prod_{\text{TERMINALS}} \Omega(\Gamma_i;t_{*}\Gamma_i)$$

$\Rightarrow$ IN PARTICULAR, THE FINITE SET OF "POLAR STATES" WHICH GOVERN ALL D4D2D0 DEGENERACIES ARE COMPUTABLE $\Rightarrow$ DERIVE OSV

B. MATH APPLICATIONS

- MODULI OF SHEAVES
- JOYCE

C. RELATION TO QUIVER GAUGE THEORY

EXAMPLE OF A:

RECALL $\Gamma = (0, P, Q, q_0) = P + Q + q_0\, dV$

REGULAR ATTRACTOR FOR $\hat{q}_0 < 0$

BLACK HOLE FOR $\hat{q}_0 \ll 0$

BUT VIEWED AS WRAPPED D4 WITH FLUX F AND N $\overline{D0}$ BRANES:

$$\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N$$

$$\leq (\hat{q}_0)_{max} = \frac{\chi(P)}{24}$$

"POLAR STATES" $0 < \hat{q}_0 < (\hat{q}_0)_{max}$

THESE MUST BE SPLIT STATES

PURE D4: $\Gamma = P + q_0\, dV$

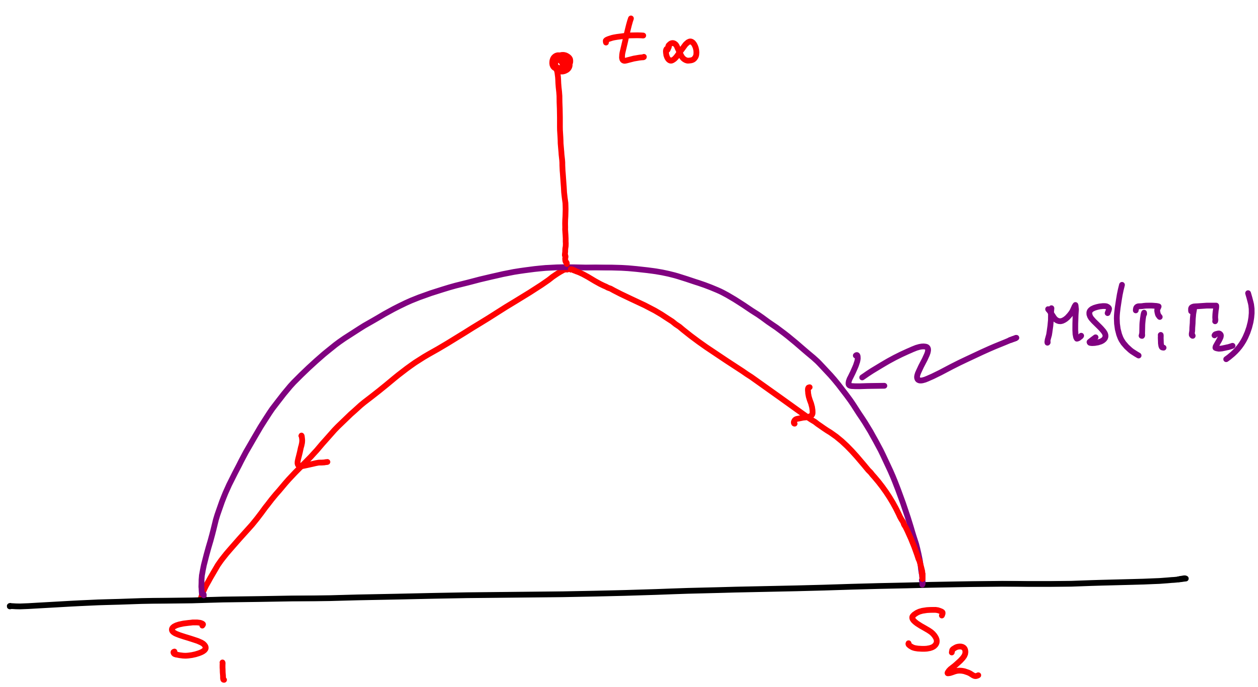
WITH $q_0 = \hat{q}_0 = (\hat{q}_0)_{max} = \frac{\chi(P)}{24}$

FIND ONLY ONE SPLITTING

$$\Gamma = P + q_0\, dV = \Gamma_1 + \Gamma_2$$

$$= \underbrace{e^{S_1}\left(1+\frac{c_2(X)}{24}\right)}_{\text{1 D6 WITH FLUX} = S_1} - \underbrace{e^{S_2}\left(1+\frac{c_2(X)}{24}\right)}_{1\,\overline{\text{D6}}\text{ W/ FLX } S_2}$$

$$S_1 - S_2 = \underline{P}$$



$$\Omega(\Gamma, t_\infty) = (-1)^{I_{12}-1}|I_{12}|\,\Omega(\Gamma_1)\,\Omega(\Gamma_2) = (-1)^{I_{12}-1}|I_{12}|$$

$$I_{12} = \langle \Gamma_1, \Gamma_2 \rangle = \frac{P^3}{6} + \frac{c_2(X)\cdot P}{12}$$

INDEED = THE CORRECT ANSWER FOR

χ (MODULI OF PURE D4)

B. MATH APPLICATIONS
(WITH E. DIACONESCU)

CONSIDER THE CASE WHERE D4 WRAPS A RIGID SURFACE S IN C.Y.

$\mathcal{P}$ IS NOT IN KÄHLER CONE

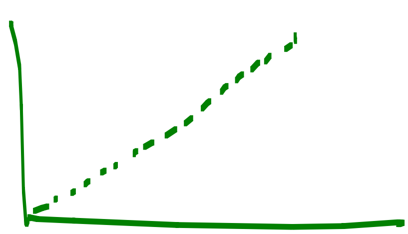
EXAMPLE $r=2$ BUNDLE ON S WITH $b^{11}(S) > 1$

AS A FUNCTION OF J CAN HAVE INSTABILITY:

$$0 \to \mathcal{I}_{Z_2}(n_2) \to \mathcal{E} \to \mathcal{I}_{Z_1}(n_1) \to 0$$

SUCH BUNDLES BECOME UNSTABLE ACROSS WALLS IN KÄHLER CONE

Exple $\mathbb{P}^1 \times \mathbb{P}^1$



REFINE $\Omega(y;\Gamma;t) = \mathrm{Tr}_{\mathcal{H}'(\Gamma;t)}(-y)^{2J_3}$

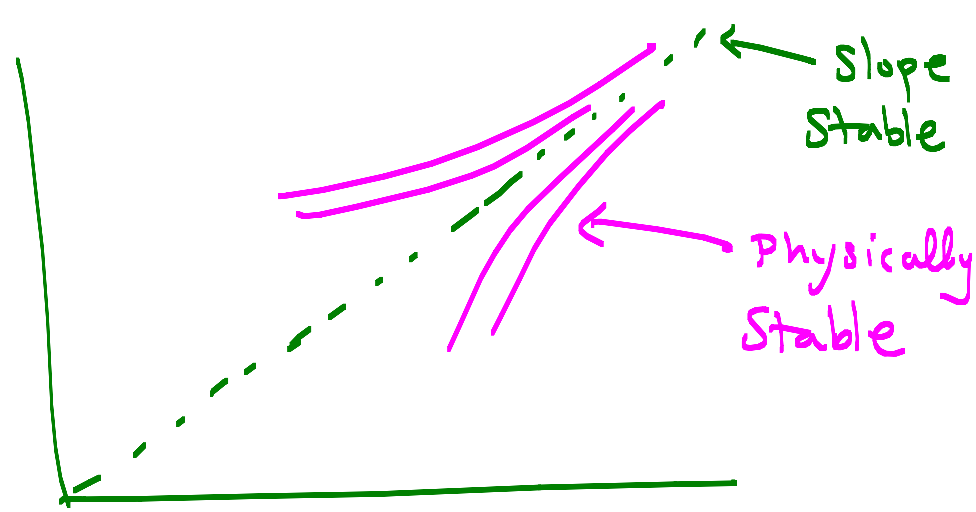
$$\mathcal{H}' \sim H^*\{\text{MODULI OF BRANES}\}$$

$\Omega(y;\Gamma;t) =$ POINCARÉ POLYNOMIAL

$$\Delta\Omega = (-y)^{-\langle\Gamma_1,\Gamma_2\rangle+1} \frac{1-y^{2\langle\Gamma_1,\Gamma_2\rangle}}{1-y^2} \Omega(y;\Gamma_1)\Omega(y;\Gamma_2)$$

$\Longrightarrow$ REPRODUCE RESULTS OF GÖTTSCHE AND YOSHIOKA ON MODULI OF BUNDLES,

MORE IMPORTANT: WALLS OF MS ASYMPTOTE TO WALL OF SLOPE STAB.



$\Longrightarrow$ MODULI OF D4

WRAPPING A RIGID HOLO. SURFACE

IS NOT MODULI SPACE OF (SLOPE STABLE)

SHEAVES!! EVEN WHEN INST.

CORRECTIONS ARE NEGLECTED.

CONTRADICTS STATEMENTS FOUND

IN THE LITERATURE. (INCLUDING MY

PAPERS)

PRESUMABLY THE RIGHT CONCEPT

IS THE MODULI SPACE OF STABLE

OBJECTS IN THE DERIVED CATEGORY

(YET TO BE CONSTRUCTED)

6. APOTHEOSIS OF DONALDSON, THOMAS & McMAHON

HALO STATES

SUPPOSE

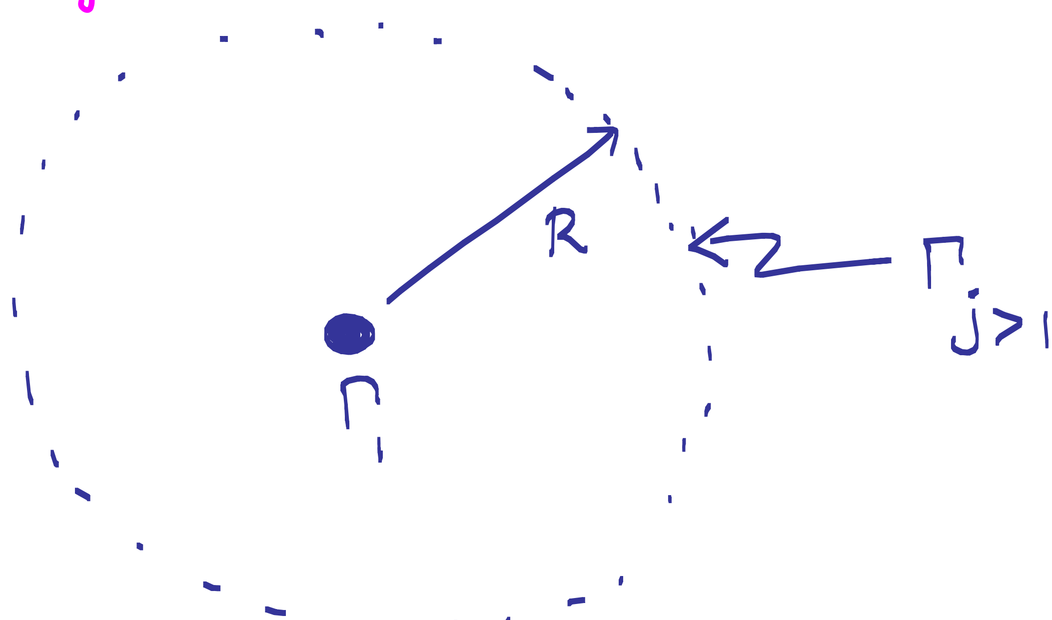
$$\Gamma_j = \lambda_j \Gamma_2 \qquad \lambda_j > 0, \; j = 2, \cdots, N$$

ARE ALL MUTUALLY LOCAL

INTEGRABILITY CONDITIONS SAY

$$j \geq 2: \quad \frac{\langle \Gamma_j, \Gamma_1 \rangle}{|\vec{x}_j - \vec{x}_1|} = 2 \frac{\mathrm{Im}\left(Z(\Gamma_j) \overline{Z(\Gamma)} \right)}{|Z(\Gamma)|}$$

$\Rightarrow$ ALL $|\vec{x}_j - \vec{x}_1|$ ARE EQUAL



IMPORTANT EXAMPLE:

D6D2D0 BOUND TO D2D0

D6D2D0 CHARGE:

$$\Gamma(\beta, n) := \Gamma = (1, 0, -\beta, n)$$

$\beta = P.D.[\sigma] \quad \sigma \subset X$ HOLOMORPHIC CURVE

Remark: Corresponds to an "IDEAL SHEAF"

D2D0 CHARGE:

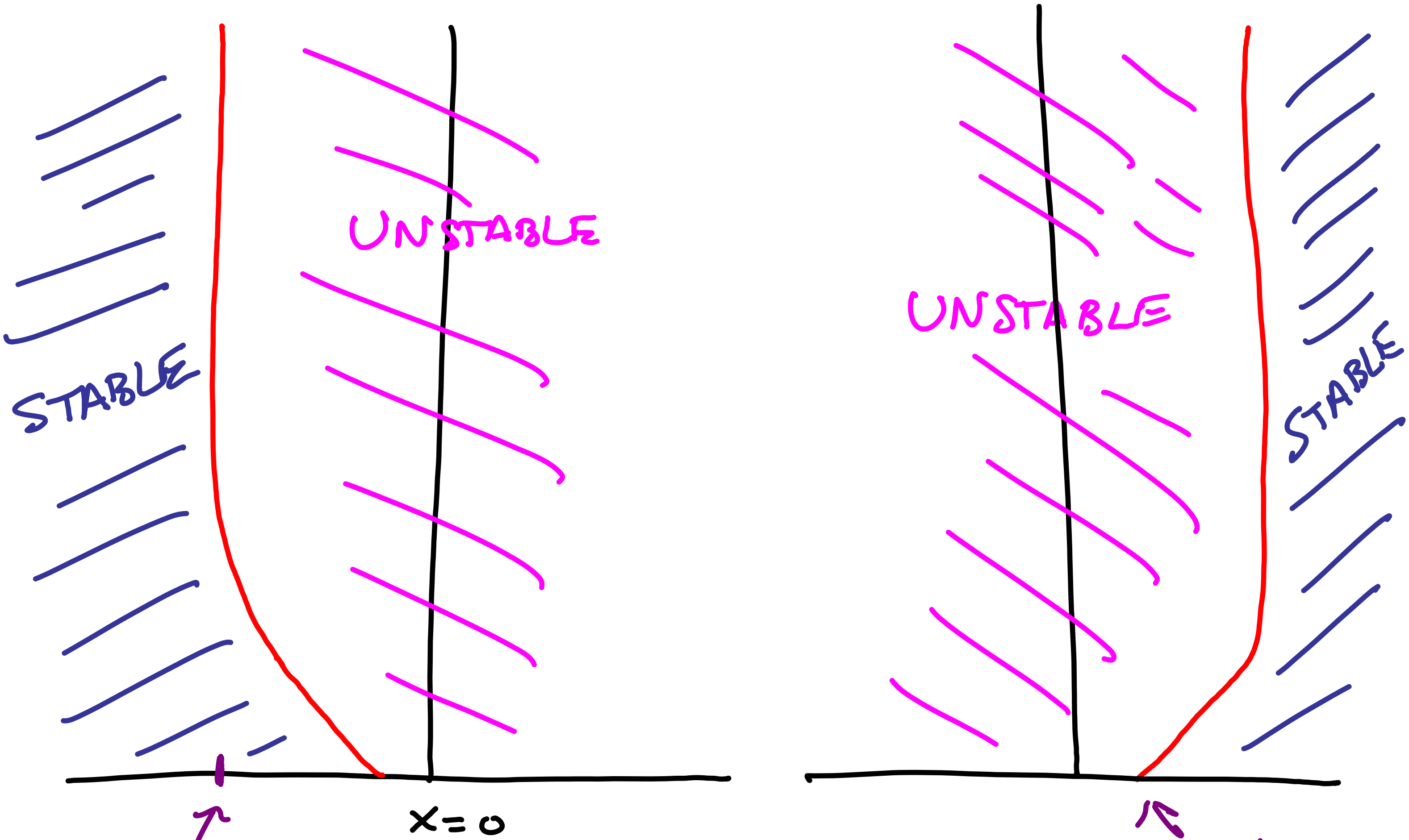
$$\Gamma_h = (0, 0, -\beta_h, n_h)$$

SET $t = (x + iy) P$

PLOT MARGINAL STABILITY CURVE

$$\operatorname{Im} Z_1 Z_h^* = 0$$

FOR $P \cdot \beta_h > 0$

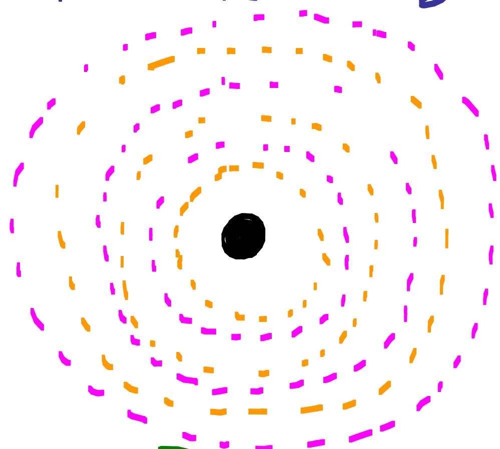


$$x = \frac{n_h}{2P_h \cdot \beta} \qquad \boxed{n_h < 0} \qquad\qquad \boxed{n_h > 0} \qquad x = \frac{n_h}{2P \cdot \beta_h}$$

- DEPENDING ON B-FIELD THERE IS A D2D0 HALO – EVEN AT $J \longrightarrow \infty$!

$\Gamma(\beta, n)$ FORM HALO BOUNDSTATES WITH D2D0 STATES PROPORTIONAL TO $(0, 0, -\beta_h, n_h)$.

GENERAL PICTURE: BOHR MODEL



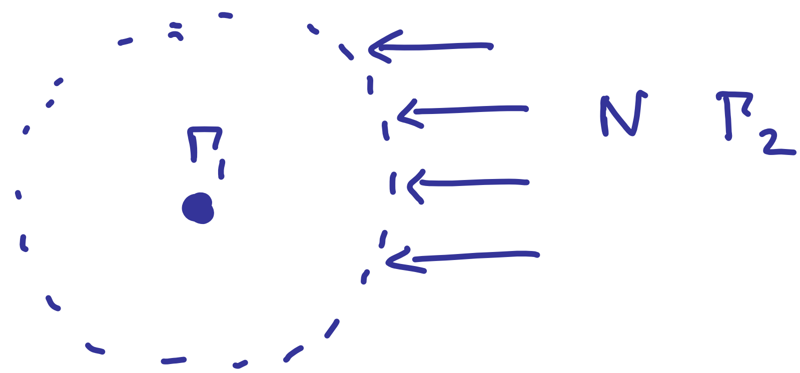
INCREASE $B = xP$, $x \to +\infty$: FOR $x > 0$ ALL $n_h < 0$ STATES HAVE DECAYED.

AS $x \to +\infty$ WE MOVE INTO THE STABLE REGION FOR ALL $n_h > 0$, AND EVER LARGER "ATOMS" BECOME STABLE

REMARK: $\Omega(\Gamma(\beta, n); t) \neq$ DONALDSON-THOMAS INVARIANTS!

NONPRIMITIVE WALL-CROSSING

$$\Gamma \longrightarrow \Gamma_1 + N\,\Gamma_2$$



THE PARTICLES IN THE HALO GENERATE A FOCK SPACE WITH

$$(J_{\Gamma_1, k\Gamma_2}) \otimes \mathcal{H}(k\Gamma_2; t_{ms})$$

CREATION/ANNIH OPERATORS OF CHARGE $k\Gamma_2$

ALL WALLS $W(\Gamma_1, N\Gamma_2)$ COINCIDE $\Longrightarrow$ CROSSING A WALL WE LOSE ENTIRE FOCK SPACE:

$$\Omega(\Gamma_1) + \sum_{N\geq 1} \Delta\Omega(\Gamma \to \Gamma_1 + N\Gamma_2)\, u^N$$

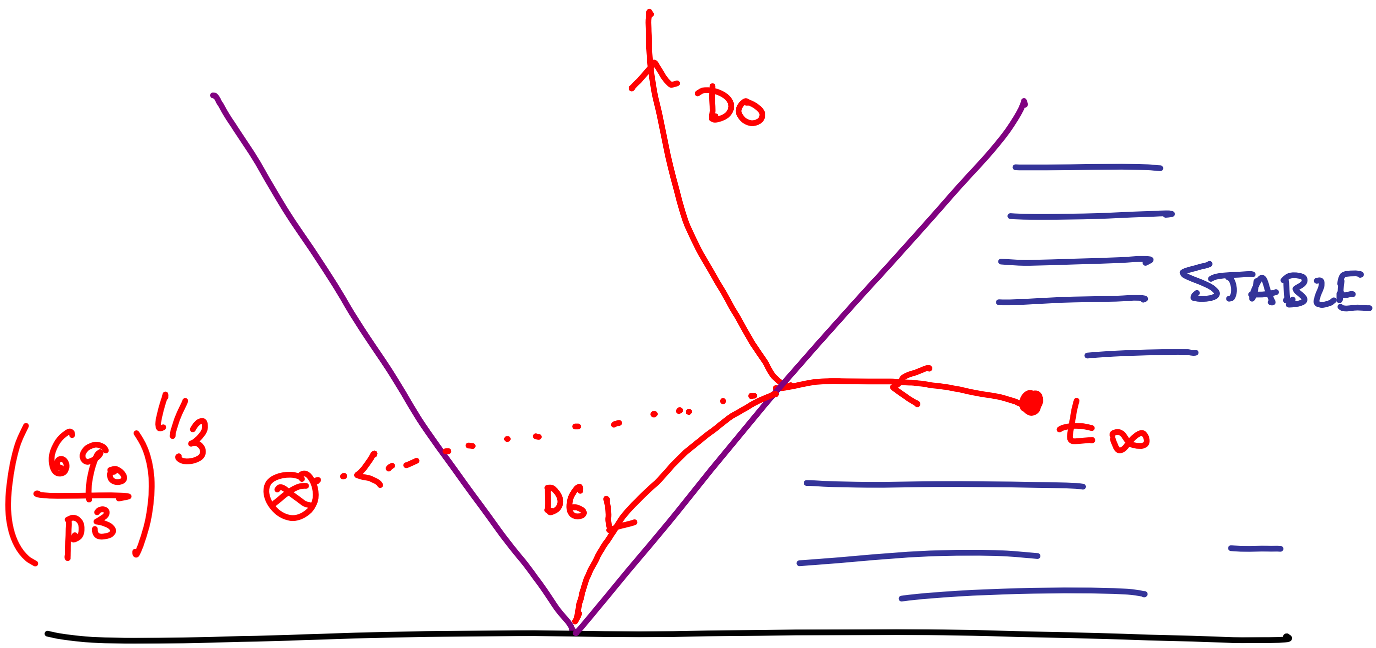
$$= \Omega(\Gamma_1) \prod_{k>0} \left(1 - (-1)^{k\langle\Gamma_1, \Gamma_2\rangle} u^k \right)^{|\langle\Gamma_1, k\Gamma_2\rangle|\, \Omega(k\Gamma_2)}$$

EXAMPLE: D6D0

$$\Gamma = 1 + q_0 dV$$

$$Z_h = \frac{t^3}{6} - q_0$$

SET $t = (x+iy)P \implies$ ZERO @ $t = \left(\frac{6q_0}{p^3}\right)^{1/3} P$



$$Z_{D6D0}(u) = \sum \Omega(1 + q_0 dV; t)\, u^{q_0}$$

$$= \begin{cases} 1 & t \in \text{UNSTABLE} \\ \prod_{k>0} \left(1 - (-u)^k\right)^{-k\chi(X)} & t \in \text{STABLE} \end{cases}$$

McMahon!

A SIMILAR DISCUSSION APPLIES TO D6D2D0 BOUNDSTATES

$\Longrightarrow$ INFINITE PRODUCT FORMULAE RELATED TO DONALDSON-THOMAS PARTITION FUNCTION:

$$Z_{D6D2D0}(u,v;t) := \sum_{n,\beta} \Omega(\Gamma(n,\beta);t)\, u^n v^\beta$$

$$\lim_{x\to+\infty} Z_{D6D2D0}(u,v;t) = Z_{DT}(u,v)$$

$$\lim_{x\to-\infty} Z_{D6D2D0}(u,v;t) = Z_{DT}(u^{-1},v)$$

7. ROUGH SKETCH OF OSV

1. FAREYTAIL:

$$Z_{D4D2D0} = \sum \text{MOD. TMNS. OF } Z^{POLAR}_{D4D2D0}$$

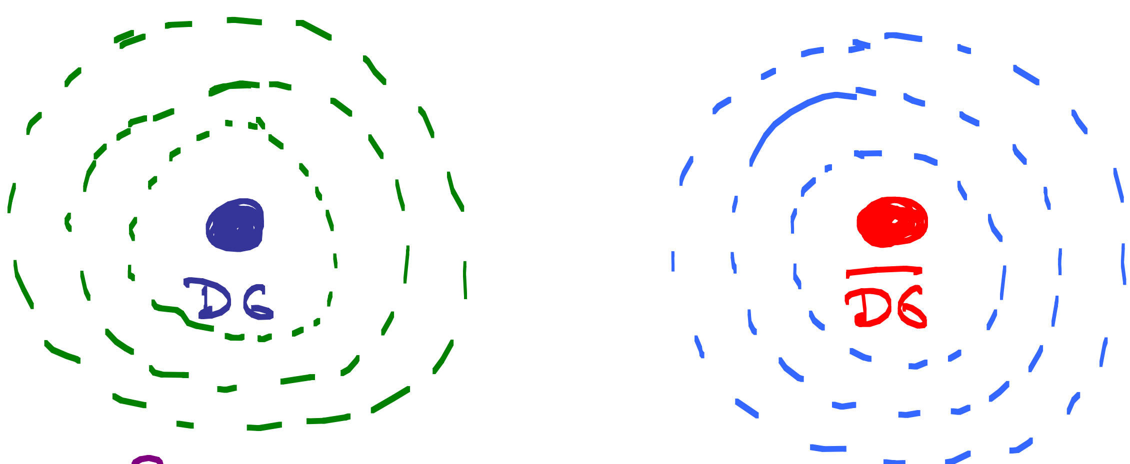
2. $$Z^{POLAR}_{D4D2D0} = Z^{\epsilon}_{D6\overline{D6}}(t_{ms}) + ET(\epsilon)$$

EPS CONJECTURE

(EXTREME) POLAR STATES SPLIT AS $D6\overline{D6}$

$$\epsilon = \frac{(\hat{q}_0)_{max} - \hat{q}_0}{(\hat{q}_0)_{max}} \ll 1 \implies$$

POLAR STATES LOOK LIKE



$$\Gamma_1 = e^{S_1}(1 - \beta_1 + n_1 dV) \qquad \Gamma_2 = -e^{S_2}(1 - \beta_2 + n_2 dV)$$

3. $$Z^{\epsilon}_{D6\overline{D6}}(t_{ms}) = Z^{\epsilon}_{D6D2D0}(t_{ms})\, Z^{\epsilon}_{\overline{D6D2D0}}(t_{ms})$$

W.C. Formula

4. $$Z^{\epsilon}_{D6D2D0}(t_{ms}) = Z^{\epsilon}_{DT}$$

$$\epsilon \sim |P|^{-\xi_{cd}} \qquad P \longrightarrow \infty$$

SWING STATE CONJECTURE: $\xi_{cd} < 2$

5. $Z_{DT} = Z_{GW} = Z_{TOP}$ MNOP CONJ.

STEPS 1 & 2 MAKE IMPORTANT APPROXIMATIONS

8. Conclusion: Some Important Open Problems

- ENTROPY ENIGMA:

  * DOES e^{λ^3} CANCEL TO e^{λ^2}?

  MANY ARG'S FOR AND AGANST

- GENERALIZE W.C. TO $N_1\Gamma_1 + N_2\Gamma_2$

- RELATION TO BPS ALGEBRAS?

- $\Omega(\Gamma; t_*(\Gamma))$ HAS NO W.C.

  BUT ALSO NO APPARENT AUTOMORPHY PROPERTIES....

COULD IT NEVERTHELESS BE THE CASE THAT THIS IS THE PROPER OBJECT TO USE IN AN OSV-LIKE FORMULA?

- PHYSICALLY $\exists$ DEF$^{\underline{N}}$ OF THE NONPTVE TOP. STRING?

  — THE $D6\overline{D6}$ PICTURE SUGGESTS A WAY TO APPROACH THIS PROBLEM.