# Quaternion-Kähler Geometry

ADVANCES IN STRING THEORY,
WALL CROSSING, AND QK GEOMETRY
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# Contents

## 1.1 The Wolf spaces

are quaternionic analogues of Hermitian symmetric spaces. The classical compact ones of real dimension $4n$ are

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$$

$$\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n)\times U(2))}$$

$$\mathbb{G}\mathrm{r}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\times SO(4)}.$$

Exceptional ones have real dimensions $8, 28, 40, 64, 112$:

$$\frac{G_2}{SO(4)},\quad \frac{F_4}{Sp(3)Sp(1)},\quad \frac{E_6}{SU(6)Sp(1)},\quad \frac{E_7}{Spin(12)\,Sp(1)},\quad \frac{E_8}{E_7\,Sp(1)}.$$

Of all these, $\mathbb{G}\mathrm{r}_2(\mathbb{C}^{n+2})$ (and $\mathbb{G}\mathrm{r}_4(\mathbb{R}^6)$) are also Kähler. The others have $b_2 = 0,$ and cannot even admit an almost complex structure (Gauduchon-Moroianu-Semmelmann).

## 1.2 Uniform construction

Given a compact simple Lie algebra $\mathfrak{g}$, choose a Lie subalgebra $\mathfrak{su}(2) = \mathfrak{sp}(1)$ arising from a highest root. Set

$$H = K\,Sp(1) = \{g \in G : \mathrm{Ad}(g)(\mathfrak{su}(2)) = \mathfrak{su}(2)\}.$$

Then

$$M = \frac{G}{K\,Sp(1)} = \frac{G}{H}$$

is quaternion-Kähler (QK); if $G$ is centreless,

$$H \subseteq Sp(n)Sp(1) \subset SO(4n),$$

where $Sp(n) = Sp(n)_\ell$ and $Sp(1) \subset Sp(n)_r$ are subgroups of $SO(4n)$ arising from left and right multiplication on $\mathbb{H}^n = \mathbb{R}^{4n}$:

$$Sp(n)Sp(1) = Sp(n) \times_{\mathbb{Z}_2} Sp(1) = Sp(n) \cdot Sp(1).$$

**Theorems:** All compact QK homogeneous spaces arise in this way. There exist homogeneous non-symmetric QK spaces with $s < 0$ (Alekseevsky).

Later we'll see what happens if we take other $\mathfrak{su}(2)$'s in $\mathfrak{g}$...

## 1.3 The isotropy representations

of these spaces have special merit. For each Wolf space $G/K\,Sp(1)$, we get a symplectic representation $K \to \mathrm{End}(\mathbb{C}^{2n})$.

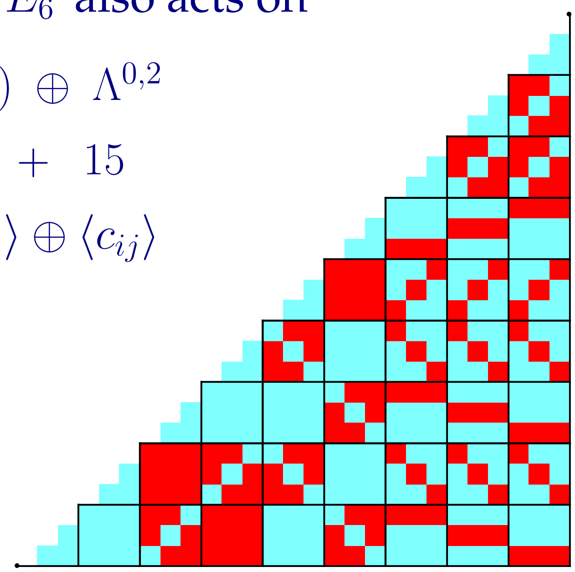
**Example:** Consider $\mathfrak{e}_6 = \mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}$, where

$$\mathfrak{m}_c = \Lambda^{3,0} \otimes \Sigma = \mathbb{C}^{40}$$

is the tangent space and $\Sigma = \mathbb{C}^2$. But $E_6$ also acts on

$$\begin{aligned} \mathbb{C}^{27} &= (\Lambda^{1,0} \otimes \Sigma) \oplus \Lambda^{0,2} \\ &= 6 + 6 + 15 \\ &= \langle a_i \rangle \oplus \langle b_j \rangle \oplus \langle c_{ij} \rangle \end{aligned}$$

giving Schläfli's configuration of the 27 lines on a cubic surface:



## 2.1 Holonomy properties

A *quaternion-Kähler manifold* is a Riemannian manifold of dim $4n$, with $n \geqslant 2$, whose holonomy group $H$ equals $Sp(n)Sp(1)$ or a subgroup thereof.

(i) Quaternion-Kähler $\not\Rightarrow$ Kähler

(ii) If $H \subsetneq Sp(n)Sp(1)$ then $M$ must be symmetric.

(iii) One normally excludes the *hyper-Kähler* (HK) case $H \subseteq Sp(n)$.

(iv) Bearing in mind that $Sp(1)Sp(1) = SO(4)$, one imposes that $M$ be *self-dual* and *Einstein* when $n=1$.

The Riemann curvature tensor $R$ of a QK manifold belongs to

$$S^2(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \cong S^2\mathfrak{sp}(n) \oplus \mathfrak{sp}(n)\mathfrak{sp}(1) \oplus S^2\mathfrak{sp}(1),$$

but most summands on the right are Bianchi-inconsistent, so

$$R = R_{\mathrm{HK}} \oplus sR_0, \qquad R_{\mathrm{HK}} \in S^2\mathfrak{sp}(n),$$

where $\mathbb{HP}^n$ has $R = sR_0$ (cf. $M^6$ nearly-Kähler).

**Corollary:** $M$ is necessarily Einstein. It is (locally) hyper-Kähler iff the scalar curvature $s$ vanishes.

## 2.2 Parallel bundles and forms

Let $M$ be a QK manifold. The reduction to $Sp(n)Sp(1)$ equips each tangent space $T_mM$ with a 2-sphere

$$Z_m = \{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}$$

of almost complex structures, where $IJ = K = -JI$. They are compatible with a Riemannian metric $g$, and the subbundle $Z$ of $\mathrm{End}(TM)$ is invariant by the Levi-Civita connection $\nabla$.

Equivalently, we have a rank 3 vector bundle with fibre

$$V_m = \{a\,\omega_1 + b\,\omega_2 + c\,\omega_3 : a, b, c \in \mathbb{R}\} \subset \Lambda^2 T_m^* M,$$

and a parallel 4-form $\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r$.

In 8 dimensions ($n = 2$), $\Omega$ is linearly equivalent to

$$1234 + 5678 + \tfrac{1}{3}(1256 + 1278 + 3456 + 3478 + 1357 + 1386 \\ +4257 + 4286 + 1458 + 1467 + 2358 + 2367).$$

Its stabilizer is $Sp(2)Sp(1)$ (but $Spin(7)$ if we change a few signs).

**Lemma:** If $n \geqslant 3$ the condition $d\Omega = 0$ implies $\nabla\Omega = 0$ and the holonomy reduction (Swann).

## 2.3 Representation theory

Suppose that $M$ has an $Sp(n)Sp(1)$ structure. Its complexified (co)tangent space is

$$(T_m^*M)_c = E \otimes H, \qquad E = \mathbb{C}^n, \quad H = \mathbb{C}^2$$

The space of 2-forms is

$$\begin{aligned}(\Lambda^2 T_m^* M)_c &\cong S^2E \oplus S^2H \oplus (\Lambda_0^2 E \otimes S^2 H)\\ &\cong \mathfrak{sp}(n)_c \oplus \mathfrak{sp}(1)_c \oplus \mathfrak{m}_c.\end{aligned}$$

Now suppose that $\dim M = 8$. Then $\mathfrak{m}$ is the tangent space to

$$\frac{SO(8)}{Sp(2)\times Sp(1)} \overset{1:2}{\longleftarrow} \frac{SO(8)}{SO(5)\times SO(3)} = \mathbb{G}\mathrm{r}_3(\mathbb{R}^8),$$

and $\nabla\Omega$ belongs to the space

$$T^*M \otimes \mathfrak{m} \;\cong\; \begin{array}{|c|c|}\hline E\,S^3H & K\,S^3H \\ \hline E\,H & K\,H \\ \hline\end{array}$$

**Example:** The 8-manifold $SU(3)$ has an $Sp(2)Sp(1)$ structure with $\nabla\Omega \in E\,S^3H$ (Macia).

## 3.1 Quaternionic manifolds

Recall that any QK manifold $M$ admits a subbundle $Z \subset \mathrm{End}(TM)$, hence a $G$-structure with $G = GL(n,\mathbb{H})Sp(1)$ and a torsion-free $G$-connection. This makes it a *quaternionic* manifold.

If we reduce to $SL(n,\mathbb{H})Sp(1)$, the forsion-free connection is *unique*.

**Theorem:** Given a quaternionic manifold with $n \geqslant 2$, the total space of $Z$ has a natural complex structure (Bérard Bergery, S).

This generalizes the twistor space construction of Atiyah-Hitchin-Singer for self-dual conformal structures.

**Example:** If the structure reduces to $GL(n,\mathbb{H})$ or $SL(n,\mathbb{H})$ then $M$ is *hyper-complex*: it admits global complex structures $I, J, K$ and there is a holomorphic map

$$Z \cong M \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1.$$

If $M = \mathbb{H}^n$ then $Z \cong 2n\mathcal{O}(1)$.

**Corollary:** Over any quaternionic manifold, we can choose a *local* basis $I, J, K$ with $I$ integrable and $IJ = K = -JI$. This makes QK manifolds very close to being complex and (if $s > 0$) Kähler.

## 3.2 Associated bundles

arise from the actions of $Sp(1)$ on model fibres:

$$
\begin{array}{ccccc}
\mathscr{S}^{4n+3} & \hookrightarrow & \mathscr{U}^{4n+4} & & \\
\downarrow & & & \searrow & \\
Z^{4n+2} & & \hookrightarrow & & V^{4n+3} \\
& \searrow & & \swarrow & \\
& & M^{4n} & &
\end{array}
$$

$M$ is a quaternionic manifold.

$Z$ is the complex twistor space with fibre $\mathbb{CP}^1 \cong S^2$.

$V$ is the span of $I, J, K,$ fibre $\mathbb{R}^3 = \mathfrak{sp}(1),$ is $\Lambda^2_+ T^*M$ if $n = 1$.

$\mathscr{U}$ is the hyper-complex Swann bundle with fibre $\mathbb{H}^*/\mathbb{Z}_2$;
$\mathscr{U}$ has both HK and QK metrics if $M$ is QK with $s > 0$.

$\mathscr{S}$ has fibre $SO(3),$ and is 3-Sasakian if $M$ is QK with $s > 0$;
$\mathscr{S}$ can be smooth even if $M$ is an orbifold.

(Roček, Boyer-Galicki)

## 3.3 Low-cost flying

over a quaternionic manifold $M^{4n}$ is achieved exploiting a notion of instanton, namely a bundle $(F, \nabla)$ with 'self-dual' curvature.

**Theorem:** If $F$ has fibre $\mathbb{H}^k$, the total space $M_F$ of $F \otimes H$ is also a quaternionic manifold of real dimension $4(n+k)$.

Proof. If $\pi\colon Z \to M,$ the instanton condition tells us that $\pi^* F$ has a holomorphic structure over the twistor space $Z$. Then the twistor space of $M_F$ is the total space of $\pi^* F \otimes \mathcal{O}(1)$ over $Z;$ since this is complex, $M$ is quaternionic.

Special cases:

(i) if $M$ is QK then $TM \cong E \otimes H$ is quaternionic, but not (?) QK.

(ii) if $F = \mathbb{H}$ then $M_F$ admits an $\mathbb{H}^*$-invariant hypercomplex structure away from its zero section and double-covers $\mathscr{U}$.

**Example:** Given $M$ QK, construct $\mathscr{U}$ which is HK. Then

$$\mathscr{U} \cong \frac{\mathscr{U} \times \mathbb{H}^*}{\mathbb{H}^*}$$

also has a QK metric!

## 4.1 Twistor spaces

The prototype is given by the fibration

$$\begin{array}{ccccc} \mathbb{CP}^3 & \supset & \mathbb{CP}^3 \setminus \mathbb{CP}^1 & \to & \mathbb{CP}^1 \\ \downarrow & & \downarrow & & \\ S^4 = \mathbb{HP}^1 & \supset & \mathbb{R}^4. & & \end{array}$$

Conformal geometry of $S^4$ is encoded into holomorphic data in $\mathbb{CP}^3$ invariant by the real involution $j$ acting as the antipodal map on each fibre $S^2 = \mathbb{CP}^1$. A (local) holomorphic section is the same as an *orthogonal complex structure* on (an open set of) $\mathbb{R}^4$.

**Applications** (S -Viaclovsky):

(i) Any OCS over $\mathbb{R}^4 \setminus \{p_1, \ldots, p_n\}$ is conformally constant.

(ii) This is false for $\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2$ that inherits an OCS from $\mathbb{CP}^3$ !

(iii) A non-real quadric in $\mathbb{CP}^3$ has at most 2 twistor lines. Any non-singular cubic surface in $\mathbb{CP}^3$ has at most $4/27$ twistor lines.

## 4.2 Fano contact manifolds

When $M^{4n}$ is a Wolf space, its twistor space

$$Z = \frac{G}{KU(1)} \stackrel{\pi}{\longrightarrow} \frac{G}{K\,Sp(1)} = M.$$

is an adjoint orbit in $\mathfrak{g}$, polarized by a holomorphic line bundle $L$. Each fibre $\pi^{-1}(m)$ is a rational curve $\mathbb{CP}^1$ with normal bundle $2n\mathcal{O}(1)$ (whereas $L|_{\mathbb{CP}^1} \cong \mathcal{O}(2)$ ).

Wolf, in 1964, pointed out that $Z$ has a complex contact structure $\theta \in H^0(Z, \Omega^1(L))$. There is a holomorphic short exact sequence

$$0 \to D \to TZ \stackrel{\theta}{\longrightarrow} L \to 0$$

of vector bundles, in which $D$ is horizontal, and $\overline{\kappa} \cong L^{n+1}$.

**Example:** $\mathbb{CP}^{2n+1}(\to \mathbb{HP}^n)$ has $L = \mathcal{O}(2)$, but in general $Z$ is Fano of index $n+1$.

## 4.3 The Penrose correspondence

between $M$ and $Z$ is more general, even in the QK context:

| $M$ QK, $s \neq 0$ | $Z$ complex contact |
|---|---|
| point | rational curve |
| complex structure | holomorphic section |
| Killing field $X$ | $s \in H^0(Z, \mathcal{O}(L))$ |
| Dirac operator | $\overline{\partial}$ on $\Lambda^{0,*} \otimes \mathcal{O}(-n)$ |
| $s > 0$ | $Z$ Kähler-Einstein |
| $s > 0$, compact | $Z$ contact Fano |
| $b_2(M) + 1$ | $= b_2(Z)$ |

The interpretation of solutions to linear field equations as elements of Čech cohomology is the essence of the Penrose programme.

**Questions:**

(i) Is every contact Fano manifold $Z^{2n+1}$ homogeneous?

(ii) Is every positive QK manifold (meaning complete with $s > 0$) symmetric? There are at most finitely many examples (LeBrun-S).

These are open if $n \geqslant 3,$ and there are many global issues if $s < 0.$

## 5.1 Isometries

**Theorem** The moduli space of complete QK metrics with $s < 0$ on $\mathbb{R}^{4n}$ is infinite-dimensional (LeBrun).

Non-compact duals of Wolf spaces furnish examples with $s < 0$. Each admits a simply transitive solvable group of isometries, but Alekseevsky found three other series of such homogeneous QK spaces (amplified by de Wit-Van Proeyen and Cortés). Links with special Kähler geometry exploited $U(n,2)/(U(n) \times U(2))$ as fibre (Ferrara-Sabharwal).

Whereas symmetric spaces admit compact quotients by a discrete group, the other homogeneous spaces definitely do not.

**Question:** Does any homogeneous QK space with $s < 0$ admit a simply transitive group of isometries?

Locally, QK metrics (with $s > 0$, $s = 0$ or $s < 0$) can easily be constructed from the quotient construction.

## 5.2 Reduction

Suppose that $M^{4n}$ is a QK manifold with an isometric $U(1)$ action generating a Killing vector field $X$ such that $\mathscr{L}_X\Omega \equiv 0$. Then the $\mathfrak{sp}(1)$ -component

$$\eta = \pi(dX^\flat) \in \Gamma(M, V)$$

is a 2-form that satisfies

$$d\eta = sX \lrcorner \Omega = s\sum_{i=1}^{3} I_i X^\flat \wedge \omega_i.$$

Moreover,

$$sX \lrcorner \eta = df, \qquad f = \tfrac{1}{2}\|\eta\|^2.$$

The 2-form $\eta$ determines a section $s_\eta \in H^0(Z, \mathcal{O}(L))$ whose zero set consists of OCS's $\pm J_\eta$ on $\widehat{M} = M \setminus \{f{=}0\}$.

**Theorem:** If $U(1)$ acts freely then $f^{-1}(0)/U(1)$ admits a natural QK structure (Galicki-Lawson).

This extends naturally to an isometric action by a Lie group $G$, and is a version of the Hyper-KähLeR quotient construction.

## 5.3 G2Q structures

develop ideas from Atiyah-Witten's paper on M-theory.

**Example:** The diagonal action of $S^1 = U(1) \subset Sp(3)$ on $\mathbb{H}^3$ gives rise to an $SU(3)$-equivariant picture

$$\begin{array}{ccc} f^{-1}(0) = S^5 & \subset & \mathbb{HP}^2 \setminus \mathbb{CP}^2 \\ \downarrow & & \downarrow \\ \mathbb{CP}^2 & \longleftarrow & \Lambda^2_- T^*\mathbb{CP}^2 = X. \end{array}$$

The 7-dim space $X$ admits a complete metric with holonomy $G_2$.

**Theorem:** Let $M^8$ be QK with an $S^1$ action. Then
(i) $\widehat{M} = M \setminus \{f = 0\}$ has an explicit Kähler metric (Haydys).
(ii) $f^{-1}(c)/S^1$ has half-flat structures (Gambioli-Nagatomo-S).
(iii) $\widehat{M}/S^1$ has a $G_2$-structure with $d\varphi \equiv 0$.

The 3-form $\varphi$ is a modification of $X \lrcorner \Omega$. Holonomy $G_2$ is achieved when the gradients of $f$ and $\|X\|$ are parallel.

## 6.1 Nilpotency

Now suppose that $M^{4n}$ is a QK manifold with an isometry group $G$ of dimension $\ell$. Consider the morphism

$$\begin{array}{rcl} \Phi \ : \ Z & \to & \mathbb{P}(\mathfrak{g}_c) = \mathbb{P}(H^0(Z, \mathcal{O}(L))) \\ z & \mapsto & [s_1(z), \ldots, s_\ell(z)], \end{array}$$

a moment map for the $G_c$-invariant contact structure $\theta$.

Suppose $\wp \in S^k\mathfrak{g}^*$ is an invariant polynomial. Then either

(a) the image of $\wp$ under $S^k\mathfrak{g}_c^* \to H^0(Z, \mathcal{O}(L^k))^*$ is non-zero, or

(b) $\Phi(Z)$ lies in the zero set of $\wp$.

In (a), the image of $\wp$ vanishes on $k$ local sections of $Z \to M$ each of which determines a *$G$-invariant* OCS of type $aI+bJ+cK$. If these are absent, (b) asserts that $\Phi(Z)$ lies in the projectivized nilpotent variety in $\mathbb{P}(\mathfrak{g}_c^*)$.

## 6.2 The Nahm calibration

Nilpotent orbits in $\mathfrak{g}_c^*$ are obtained by choosing $\mathfrak{su}(2) \subset \mathfrak{g}$ :

$$\mathscr{U} = (\mathrm{Ad}\, G_c)(e) \subset \mathfrak{g}_c, \qquad e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2, \mathbb{C}).$$

Such orbits admit a HK metric (Kronheimer, Biquard), but only if $\mathscr{U}$ is minimal is $\mathscr{U}/\mathbb{C}^*$ compact . In this case, it is the twistor space $Z$ that fibres over $M = G/K\,Sp(1)$.

The fundamental 3-form $\langle [X, Y], Z\rangle$ on a compact Lie algebra $\mathfrak{g}$ defines a function $f : \mathbb{G}\mathrm{r}_3(\mathfrak{g}) \to \mathbb{R}$ for which

(i) $V \in \mathbb{G}\mathrm{r}_3(\mathfrak{g})$ is critical iff $V$ is a subalgebra;

(ii) $f$ achieves its max on the Wolf space of minimal $\mathfrak{su}(2)$'s;

(iii) one can easily compute $\mathrm{Hess}(f)$ at any critical $V$.

**Example:** For $G = SU(3)$ there are two inequivalent TDS's:

$$\mathfrak{su}(2) \;\subset\; \mathfrak{su}(3), \qquad G(V) = \frac{\dot{S}U(3)}{U(2)} = \mathbb{CP}^2$$

$$\mathfrak{so}(3) \;\subset\; \mathfrak{su}(3), \qquad G(V) = \frac{\dot{S}U(3)}{SO(3)} = C^5.$$

In the second case, $\mathfrak{su}(3)_c \cong \Sigma^2 \oplus \Sigma^4$, where $\Sigma^q = S^q(\mathbb{C}^2)$, and

$$T_V \mathbb{G}\mathrm{r}_3(\mathfrak{g}) \;\cong\; V \otimes V^\perp \;\cong\; \Sigma^2 \otimes \Sigma^4 \;\cong\; \underset{+}{\Sigma^2} \oplus \underset{0}{\Sigma^4} \oplus \underset{-}{\Sigma^6}$$

## 6.3 SU(3) continued

$$T_V\mathbb{G}\mathrm{r}_3(\mathfrak{g}) \;\cong\; V\otimes V^\perp \;\cong\; \Sigma^2\otimes\Sigma^4 \;\cong\; \underset{+}{\Sigma^2}\oplus\underset{0}{\Sigma^4}\oplus\underset{-}{\Sigma^6}$$

The associated *unstable manifold* $M^8$ is the union of $C^5$ and the upward flow lines of the vector field $\mathrm{grad}\, f$. It is diffeomorphic to a rank 3 vector bundle over $C^5$ with fibre $\Sigma^2$, and

$$T_cM^8 \cong \Sigma^2\oplus\Sigma^4 \cong \Sigma^3\otimes\Sigma^1 = \Sigma^3\otimes H$$

is quaternionic. In fact, $M^8$ is locally symmetric:

$$\frac{G_2}{SO(4)}\setminus\mathbb{CP}^2 \xrightarrow{3:1} M^8.$$

**Theorem** For $G$ compact simple, $f$ is a Morse-Bott function on $\mathbb{G}\mathrm{r}_3(\mathfrak{g})$. The unstable manifold determined by a TDS $V$ is QK and its Swann bundle is the associated complex nilpotent orbit $\mathscr{U}$.

## 7.1 Index theory

Let $M^{4n}$ be a Wolf space or a positive QK manifold with isometry group $G$. Its virtual $Spin(4n)$ representation is

$$\Delta_+ - \Delta_- = \Lambda_0^n(E-H) = \bigoplus_{p+q=n} (-1)^p R^{p,q},$$

where $R^{p,q} = \Lambda_0^p E \otimes S^q H.$

The coupled Dirac operator

$$\Gamma(M, \Delta_+ \otimes R^{p,q}) \longrightarrow \Gamma(M, \Delta_- \otimes R^{p,q})$$

has index $i^{p,q} = \int_M \mathrm{ch}(R^{p,q})\hat{A}(M)$. The following is an expression of Witten rigidity and a $G$*-equivariant statement*:

**Theorem:** $$(-1)^p i^{p,q} = \begin{cases} 0 & \text{if } p+q < n, \\ b_{2p-2} + b_{2p} & \text{if } p+q = n, \\ \dim G & p=0,\ q=n+2. \end{cases}$$

## 7.2 Application to dimension 8

Index theory (and the $\gamma$ filtration) gives a linear constraint on the Betti numbers and estimates on the isometry group, in terms of characteristic classes including $u \in H^4(M,\mathbb{Z})$ that represents $\Omega$.

**Example:** If $\dim M = 8$ then $b_2 + 1 = b_4$, suggesting $b_2 = 0$ or $1$. Moreover

$$\dim G = 5 + \int_M u^2.$$

If $b_4 = 1$ then

$$\dim G = \begin{cases} 5+16 & = \dim Sp(3), \\ 5+9 & = \dim G_2, \\ 5+4 & = \dim Sp(1)^3, \\ 5+1 & = \dim SO(4), \end{cases}$$

corresponding to

$$\mathbb{HP}^2 = \frac{Sp(3)}{Sp(2)\times Sp(1)}, \qquad \frac{G_2}{SO(4)}, \qquad \frac{\mathbb{HP}^2}{(\mathbb{Z}_2)^2}, \qquad ?$$

Only the first two spaces are non-singular.

## 7.3 Towards a classification

Let $M^{4n}$ be a compact positive QK manifold.

The odd Betti numbers $b_{2p+1}$ of $M^{4n}$ all vanish.

**Theorem:** If $b_2(M) > 0$ then $M$ is isometric to the Wolf space $\mathbb{Gr}_2(\mathbb{C}^{n+2})$ (LeBrun-S, Wisniewski).

A proof uses Mori theory on the twistor space. If $b_2(Z) > 1$ there exists a second family of rational curves on $Z$ transverse to the fibres over $M$, and a Fano contraction $Z \to \mathbb{CP}^{n+1}$ with *its* fibres tangent to the contact distribution $D$. This forces $Z = \mathbb{P}(T^*\mathbb{CP}^{n+1})$.

If we ignore $\mathbb{HP}^n$ then $M$ is *spin* iff $n$ is even, giving a dichotomy according to the parity of $n$.

## 7.4 The Â genus

Let $M^{4n}$ be a compact positive QK manifold. If $n$ is even, $M$ is spin and $\hat{A}(M)=0$ because $s>0$.

**Theorem** A positive QK manifold $M^8$ is isometric to a Wolf space (Poon-S).

An attempt to push this to dimension 12 using elliptic genera needs the assumption $\hat{A}(M)=0$ (Herrera-Herrera).

**Theorem:** If $b_4=1$ and $3\leqslant n\leqslant 6$ then $M\cong\mathbb{HP}^n$ (Amann).

All exceptional Wolf spaces have $b_4=1$, including $\dfrac{F_4}{Sp(3)Sp(1)}$.

**Theorem** If $n=5$ and $\hat{A}(M)=0$ then $\dim G\geqslant 15$ and $M$ is a Wolf space if (for example) $\int_M u^5>384$ (Amann).

**Question:** Does a positive QK manifold $M^{4n}$ necessarily admit an isometry group of positive dimension? Yes, at least if $n\leqslant 4$. If $n$ is odd, must $\hat{A}(M)$ vanish?

## 8.1 More topology

Consider the Poincaré polynomial

$$P(t) = 1 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots$$

of an oriented compact manifold with $\chi = P(-1) \neq 0$. Then

$$\log P(t-1) = \log \chi \; - d\,t + \phi\, t^2 + \;\cdots$$

where $d = \dim M$, and $16\phi = 4P''(-1)/\chi \; - d^2$. By construction, this coefficient is additive for products:

$$\phi(M \times N) = \phi(M) + \phi(N).$$

**Theorems:** (i) If $M^{4n}$ is compact HK then $\chi = 0$ or $\phi = -\frac{5}{6}n$.

(ii) If $M^d = G/H$ is an irreducible compact symmetric space of type ADE or any Hermitian symmetric space, $\phi = \frac{1}{12}(h(\mathfrak{g}) - 2)d$, where $h(\mathfrak{g})$ is the Coxeter number (Fino-S).

(iii) If $M^{4n}$ is an ADE Wolf space then $\phi = \frac{1}{3}n^2$.

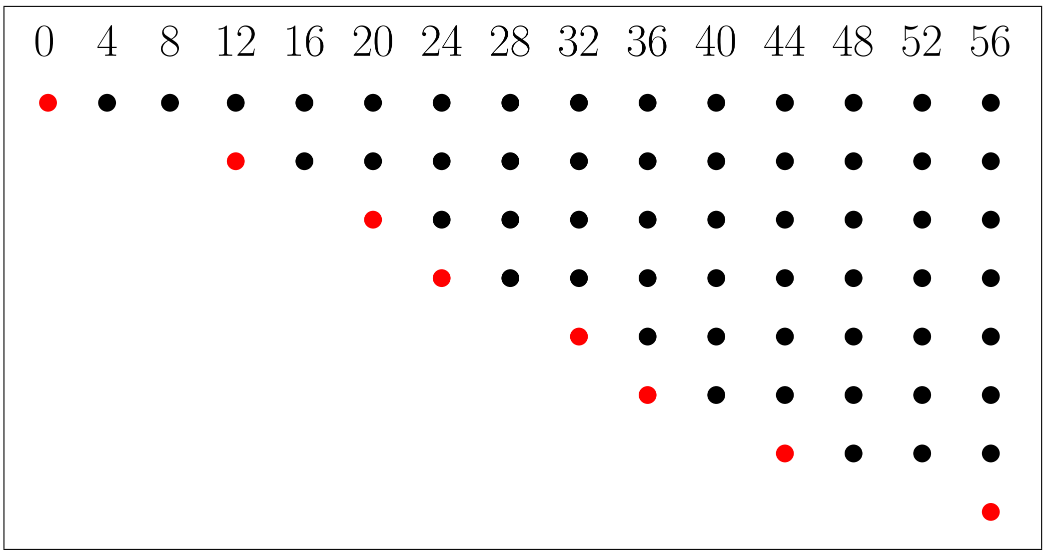
## 8.2 The case of $E_8$

The signature of an ADE Wolf space equals its rank: $b_{2n}^{+} = b_{2n} = r$, and $\chi$ equals the number $\frac{1}{2}(\dim G - r)$ of positive roots.

$E_8/E_7Sp(1)$ has 8 primitive cohomology classes $\sigma_k \in H^{4k}(M,\mathbb{R})$;

$$H^{56}(M,\mathbb{R}) = \left\langle \sigma_k \cup u^{14-k} : k = 0,3,5,6,8,9,11,14 \right\rangle,$$

exhibiting a remarkable symmetry about degree $n = 28$:




**Question:** Is the intersection form $H^{56}(M,\mathbb{Z}) \times H^{56}(M,\mathbb{Z}) \to \mathbb{Z}$ diagonalizable or the $E_8$ lattice? (Hirzebruch-Sladowy)

The quaternionic volume (Herrera, Weingart) is:

$$\int_M u^{28} = 2^3.3^2.5^2.7.31.37.41.43.47.53 = \frac{5!\,9!\,57!}{19!23!29!} = 63468758442600$$