Quaternions-Kähler Geometry

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1.1 The Wolf spaces

are quaternionic analogues of Hermitian symmetric spaces. The classical compact ones of real dimension $4n$ are

$$\mathbb{HP}^n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)}$$

$$\text{Gr}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))}$$

$$\text{Gr}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)}.$$

Exceptional ones have real dimensions $8, 28, 40, 64, 112$:

$$\frac{G_2}{SO(4)}, \frac{F_4}{Sp(3)Sp(1)}, \frac{E_6}{SU(6)Sp(1)}, \frac{E_7}{Spin(12)Sp(1)}, \frac{E_8}{E_7Sp(1)}.$$

Of all these, $\text{Gr}_2(\mathbb{C}^{n+2})$ (and $\text{Gr}_4(\mathbb{R}^6)$) are also Kähler. The others have $b_2 = 0$, and cannot even admit an almost complex structure (Gauduchon-Moroianu-Semmelmann).
1.2 Uniform construction

Given a compact simple Lie algebra \( g \), choose a Lie subalgebra \( \mathfrak{su}(2) = \mathfrak{sp}(1) \) arising from a highest root. Set

\[
H = K \mathfrak{Sp}(1) = \{ g \in G : \text{Ad}(g)(\mathfrak{su}(2)) = \mathfrak{su}(2) \}.
\]

Then

\[
M = \frac{G}{K \mathfrak{Sp}(1)} = \frac{G}{H}
\]

is quaternion-Kähler (QK); if \( G \) is centreless,

\[
H \subseteq \mathfrak{Sp}(n) \mathfrak{Sp}(1) \subset SO(4n),
\]

where \( \mathfrak{Sp}(n) = \mathfrak{Sp}(n)_\ell \) and \( \mathfrak{Sp}(1) \subset \mathfrak{Sp}(n)_r \) are subgroups of \( SO(4n) \) arising from left and right multiplication on \( \mathbb{H}^n = \mathbb{R}^{4n} \):

\[
\mathfrak{Sp}(n) \mathfrak{Sp}(1) = \mathfrak{Sp}(n) \times_{\mathbb{Z}_2} \mathfrak{Sp}(1) = \mathfrak{Sp}(n) \cdot \mathfrak{Sp}(1).
\]

**Theorems:** All compact QK homogeneous spaces arise in this way. There exist homogeneous non-symmetric QK spaces with \( s < 0 \) (Alekseevsky).

Later we’ll see what happens if we take other \( \mathfrak{su}(2) \)'s in \( g \ldots \)
1.3 The isotropy representations

of these spaces have special merit. For each Wolf space $G/KSp(1)$, we get a symplectic representation $K \rightarrow \text{End}(\mathbb{C}^{2n})$.

**Example:** Consider $\mathfrak{e}_6 = \mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{m}$, where

$$m_c = \Lambda^{3,0} \otimes \Sigma = \mathbb{C}^{40}$$

is the tangent space and $\Sigma = \mathbb{C}^2$. But $E_6$ also acts on

$$\mathbb{C}^{27} = (\Lambda^{1,0} \otimes \Sigma) \oplus \Lambda^{0,2}$$

$$= 6 + 6 + 15$$

$$= \langle a_i \rangle \oplus \langle b_j \rangle \oplus \langle c_{ij} \rangle$$

giving Schl"afli’s configuration of the 27 lines on a cubic surface:
2.1 Holonomy properties

A quaternion-Kähler manifold is a Riemannian manifold of dim $4n$, with $n \geq 2$, whose holonomy group $H$ equals $Sp(n)Sp(1)$ or a subgroup thereof.

(i) Quaternion-Kähler $\nrightarrow$ Kähler

(ii) If $H \subsetneq Sp(n)Sp(1)$ then $M$ must be symmetric.

(iii) One normally excludes the hyper-Kähler (HK) case $H \subseteq Sp(n)$.

(iv) Bearing in mind that $Sp(1)Sp(1) = SO(4)$, one imposes that $M$ be self-dual and Einstein when $n = 1$.

The Riemann curvature tensor $R$ of a QK manifold belongs to

$$S^2(sp(n) \oplus sp(1)) \cong S^2sp(n) \oplus sp(n)sp(1) \oplus S^2sp(1),$$

but most summands on the right are Bianchi-inconsistent, so

$$R = R_{HK} \oplus sR_0,$$

where $\mathbb{H}P^n$ has $R = sR_0$ (cf. $M^6$ nearly-Kähler).

**Corollary:** $M$ is necessarily Einstein. It is (locally) hyper-Kähler iff the scalar curvature $s$ vanishes.
2.2 Parallel bundles and forms

Let $M$ be a QK manifold. The reduction to $Sp(n)Sp(1)$ equips each tangent space $T_m M$ with a 2-sphere

$$Z_m = \{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}$$

of almost complex structures, where $IJ = K = -JI$. They are compatible with a Riemannian metric $g$, and the subbundle $Z$ of $\text{End}(TM)$ is invariant by the Levi-Civita connection $\nabla$.

Equivalently, we have a rank 3 vector bundle with fibre

$$V_m = \{a \omega_1 + b \omega_2 + c \omega_3 : a, b, c \in \mathbb{R}\} \subset \Lambda^2 T^* m M,$$

and a parallel 4-form $\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r$.

In 8 dimensions ($n = 2$), $\Omega$ is linearly equivalent to

$$1234 + 5678 + \frac{1}{3}(1256 + 1278 + 3456 + 3478 + 1357 + 1386 + 4257 + 4286 + 1458 + 1467 + 2358 + 2367).$$

Its stabilizer is $Sp(2)Sp(1)$ (but $Spin(7)$ if we change a few signs).

**Lemma:** If $n \geq 3$ the condition $d\Omega = 0$ implies $\nabla \Omega = 0$ and the holonomy reduction (Swann).
2.3 Representation theory

Suppose that $M$ has an $Sp(n)Sp(1)$ structure. Its complexified (co)tangent space is

$$(T_m^*M)_c = E \otimes H, \quad E = \mathbb{C}^n, \quad H = \mathbb{C}^2$$

The space of 2-forms is

$$(\Lambda^2 T_m^*M)_c \cong S^2 E \oplus S^2 H \oplus (\Lambda^2_0 E \otimes S^2 H) \cong \mathfrak{sp}(n)_c \oplus \mathfrak{sp}(1)_c \oplus \mathfrak{m}_c.$$ 

Now suppose that $\dim M = 8$. Then $\mathfrak{m}$ is the tangent space to

$$\frac{SO(8)}{Sp(2) \times Sp(1)} \xleftarrow{1:2} \frac{SO(8)}{SO(5) \times SO(3)} = \text{Gr}_3(\mathbb{R}^8),$$

and $\nabla \Omega$ belongs to the space

$$T^*M \otimes \mathfrak{m} \cong \begin{array}{|c|c|} 
E & S^3 H \\
S^3 H & K S^3 H \\
E H & K H \\
\end{array}$$

**Example:** The 8-manifold $SU(3)$ has an $Sp(2)Sp(1)$ structure with $\nabla \Omega \in E S^3 H$ (Macia).
3.1 Quaternionic manifolds

Recall that any QK manifold $M$ admits a subbundle $Z \subset \text{End}(TM)$, hence a $G$-structure with $G = GL(n, \mathbb{H})Sp(1)$ and a torsion-free $G$-connection. This makes it a quaternionic manifold.

If we reduce to $SL(n, \mathbb{H})Sp(1)$, the torsion-free connection is unique.

**Theorem:** Given a quaternionic manifold with $n \geq 2$, the total space of $Z$ has a natural complex structure (Bérard Bergery, S).

This generalizes the twistor space construction of Atiyah-Hitchin-Singer for self-dual conformal structures.

**Example:** If the structure reduces to $GL(n, \mathbb{H})$ or $SL(n, \mathbb{H})$ then $M$ is hyper-complex: it admits global complex structures $I, J, K$ and there is a holomorphic map

$$Z \cong M \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1.$$ 

If $M = \mathbb{H}^n$ then $Z \cong 2n\mathcal{O}(1)$.

**Corollary:** Over any quaternionic manifold, we can choose a local basis $I, J, K$ with $I$ integrable and $IJ = K = -JI$. This makes QK manifolds very close to being complex and (if $s > 0$) Kähler.
3.2 Associated bundles

arise from the actions of $Sp(1)$ on model fibres:

\[
\begin{array}{c}
\mathcal{S}^{4n+3} & \hookrightarrow & \mathcal{U}^{4n+4} \\
\downarrow & & \downarrow \\
\mathcal{Z}^{4n+2} & \hookrightarrow & \mathcal{V}^{4n+3} \\
\downarrow & & \downarrow \\
\mathcal{M}^{4n}
\end{array}
\]

$M$ is a quaternionic manifold.

$Z$ is the complex twistor space with fibre $\mathbb{C}P^1 \cong S^2$.

$V$ is the span of $I, J, K$, fibre $\mathbb{R}^3 = sp(1)$, is $\Lambda^2 T^* M$ if $n = 1$.

$\mathcal{U}$ is the hyper-complex Swann bundle with fibre $\mathbb{H}^*/\mathbb{Z}_2$;
$\mathcal{U}$ has both HK and QK metrics if $M$ is QK with $s > 0$.

$\mathcal{S}$ has fibre $SO(3)$, and is 3-Sasakian if $M$ is QK with $s > 0$;
$\mathcal{S}$ can be smooth even if $M$ is an orbifold.

(Roček, Boyer-Galicki)
3.3 Low-cost flying

over a quaternionic manifold \( M^{4n} \) is achieved exploiting a notion of instanton, namely a bundle \((F, \nabla)\) with ‘self-dual’ curvature.

**Theorem:** If \( F \) has fibre \( \mathbb{H}^k \), the total space \( M_F \) of \( F \otimes H \) is also a quaternionic manifold of real dimension \( 4(n + k) \).

Proof. If \( \pi: Z \to M \), the instanton condition tells us that \( \pi^* F \) has a holomorphic structure over the twistor space \( Z \). Then the twistor space of \( M_F \) is the total space of \( \pi^* F \otimes \mathcal{O}(1) \) over \( Z \); since this is complex, \( M \) is quaternionic.

Special cases:
(i) if \( M \) is QK then \( TM \cong E \otimes H \) is quaternionic, but not (?) QK.
(ii) if \( F = \mathbb{H} \) then \( M_F \) admits an \( \mathbb{H}^* \)-invariant hypercomplex structure away from its zero section and double-covers \( \mathcal{U} \).

**Example:** Given \( M \) QK, construct \( \mathcal{U} \) which is HK. Then

\[
\mathcal{U} \cong \mathcal{U} \times \mathbb{H}^* / \mathbb{H}^*
\]

also has a QK metric!
4.1 Twistor spaces

The prototype is given by the fibration

\[
\begin{align*}
\mathbb{C}P^3 \supset \mathbb{C}P^3 \setminus \mathbb{C}P^1 & \to \mathbb{C}P^1 \\
\downarrow & \\
S^4 = \mathbb{H}P^1 & \supset \mathbb{R}^4.
\end{align*}
\]

Conformal geometry of $S^4$ is encoded into holomorphic data in $\mathbb{C}P^3$ invariant by the real involution $j$ acting as the antipodal map on each fibre $S^2 = \mathbb{C}P^1$. A (local) holomorphic section is the same as an orthogonal complex structure on (an open set of) $\mathbb{R}^4$.

**Applications (S-Viaclovsky):**

(i) Any OCS over $\mathbb{R}^4 \setminus \{p_1, \ldots, p_n\}$ is conformally constant.

(ii) This is false for $\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2$ that inherits an OCS from $\mathbb{C}P^3$!

(iii) A non-real quadric in $\mathbb{C}P^3$ has at most 2 twistor lines. Any non-singular cubic surface in $\mathbb{C}P^3$ has at most $4/27$ twistor lines.
4.2 Fano contact manifolds

When $M^{4n}$ is a Wolf space, its twistor space

$$Z = \frac{G}{KU(1)} \xrightarrow{\pi} \frac{G}{KSp(1)} = M.$$ 

is an adjoint orbit in $\mathfrak{g}$, polarized by a holomorphic line bundle $L$. Each fibre $\pi^{-1}(m)$ is a rational curve $\mathbb{CP}^1$ with normal bundle $2n\mathcal{O}(1)$ (whereas $L|_{\mathbb{CP}^1} \cong \mathcal{O}(2)$).

Wolf, in 1964, pointed out that $Z$ has a complex contact structure $\theta \in H^0(Z, \Omega^1(L))$. There is a holomorphic short exact sequence

$$0 \to D \to TZ \xrightarrow{\theta} L \to 0$$

of vector bundles, in which $D$ is horizontal, and $\kappa \cong L^{n+1}$.

**Example:** $\mathbb{CP}^{2n+1}(\to \mathbb{HP}^n)$ has $L=\mathcal{O}(2)$, but in general $Z$ is Fano of index $n+1$. 
4.3 The Penrose correspondence

between $M$ and $Z$ is more general, even in the QK context:

<table>
<thead>
<tr>
<th>$M$ QK, $s \neq 0$</th>
<th>$Z$ complex contact</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>rational curve</td>
</tr>
<tr>
<td>complex structure</td>
<td>holomorphic section</td>
</tr>
<tr>
<td>Killing field $X$</td>
<td>$s \in H^0(Z, \mathcal{O}(L))$</td>
</tr>
<tr>
<td>Dirac operator</td>
<td>$\overline{\partial}$ on $\Lambda^{0,*} \otimes \mathcal{O}(-n)$</td>
</tr>
<tr>
<td>$s &gt; 0$</td>
<td>$Z$ Kähler-Einstein</td>
</tr>
<tr>
<td>$s &gt; 0$, compact</td>
<td>$Z$ contact Fano</td>
</tr>
<tr>
<td>$b_2(M) + 1$</td>
<td>$= b_2(Z)$</td>
</tr>
</tbody>
</table>

The interpretation of solutions to linear field equations as elements of Čech cohomology is the essence of the Penrose programme.

**Questions:**

(i) Is every contact Fano manifold $Z^{2n+1}$ homogeneous?

(ii) Is every positive QK manifold (meaning complete with $s > 0$) symmetric? There are at most finitely many examples (LeBrun-S). These are open if $n \geq 3$, and there are many global issues if $s < 0$. 
5.1 Isometries

**Theorem** The moduli space of complete QK metrics with \( s < 0 \) on \( \mathbb{R}^{4n} \) is infinite-dimensional (LeBrun).

Non-compact duals of Wolf spaces furnish examples with \( s < 0 \). Each admits a simply transitive solvable group of isometries, but Alekseevsky found three other series of such homogeneous QK spaces (amplified by de Wit-Van Proeyen and Cortés). Links with special Kähler geometry exploited \( U(n, 2)/(U(n) \times U(2)) \) as fibre (Ferrara-Sabharwal).

Whereas symmetric spaces admit compact quotients by a discrete group, the other homogeneous spaces definitely do not.

**Question:** Does any homogeneous QK space with \( s < 0 \) admit a simply transitive group of isometries?

Locally, QK metrics (with \( s > 0 \), \( s = 0 \) or \( s < 0 \)) can easily be constructed from the quotient construction.
5.2 Reduction

Suppose that $M^{4n}$ is a QK manifold with an isometric $U(1)$ action generating a Killing vector field $X$ such that $\mathcal{L}_X \Omega \equiv 0$. Then the $\mathfrak{sp}(1)$-component

$$\eta = \pi(dX^b) \in \Gamma(M,V)$$

is a 2-form that satisfies

$$d\eta = sX \lrcorner \Omega = s \sum_{i=1}^3 I_i X^b \wedge \omega_i.$$ 

Moreover,

$$sX \lrcorner \eta = df, \quad f = \frac{1}{2} \|\eta\|^2.$$ 

The 2-form $\eta$ determines a section $s_\eta \in H^0(Z, O(L))$ whose zero set consists of OCS’s $\pm J_\eta$ on $\widehat{M} = M \setminus \{f = 0\}$.

**Theorem:** If $U(1)$ acts freely then $f^{-1}(0)/U(1)$ admits a natural QK structure (Galicki-Lawson).

This extends naturally to an isometric action by a Lie group $G$, and is a version of the Hyper-Kähler quotient construction.
5.3 G2Q structures

develop ideas from Atiyah-Witten’s paper on M-theory.

**Example:** The diagonal action of $S^1 = U(1) \subset Sp(3)$ on $\mathbb{H}^3$ gives rise to an $SU(3)$-equivariant picture

$$f^{-1}(0) = S^5 \subset \mathbb{H}P^2 \setminus \mathbb{C}P^2$$

\[\downarrow\quad \downarrow\]

$\mathbb{C}P^2 \leftarrow \Lambda^2 T^* \mathbb{C}P^2 = X$.

The 7-dim space $X$ admits a complete metric with holonomy $G_2$.

**Theorem:** Let $M^8$ be QK with an $S^1$ action. Then

(i) $\hat{M} = M \setminus \{f = 0\}$ has an explicit Kähler metric (Haydys).

(ii) $f^{-1}(c)/S^1$ has half-flat structures (Gambioli-Nagatomo-S).

(iii) $\hat{M}/S^1$ has a $G_2$-structure with $d\varphi \equiv 0$.

The 3-form $\varphi$ is a modification of $X \ominus \Omega$. Holonomy $G_2$ is achieved when the gradients of $f$ and $\|X\|$ are parallel.
6.1 Nilpotency

Now suppose that $M^{4n}$ is a QK manifold with an isometry group $G$ of dimension $\ell$. Consider the morphism

$$\Phi : Z \to \mathbb{P}(g^*_c) = \mathbb{P}(H^0(Z, O(L)))$$

$$z \mapsto [s_1(z), \ldots, s_\ell(z)],$$

a moment map for the $G_c$-invariant contact structure $\theta$.

Suppose $\varphi \in S^k g^*$ is an invariant polynomial. Then either

(a) the image of $\varphi$ under $S^k g^*_c \to H^0(Z, O(L^k))^*$ is non-zero, or

(b) $\Phi(Z)$ lies in the zero set of $\varphi$.

In (a), the image of $\varphi$ vanishes on $k$ local sections of $Z \to M$ each of which determines a $G$-invariant OCS of type $aI + bJ + cK$. If these are absent, (b) asserts that $\Phi(Z)$ lies in the projectivized nilpotent variety in $\mathbb{P}(g^*_c)$. 
6.2 The Nahm calibration

Nilpotent orbits in $g_c^*$ are obtained by choosing $su(2) \subset g$:

$$\mathcal{U} = (Ad \, G_c)(e) \subset g_c, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

Such orbits admit a HK metric (Kronheimer, Biquard), but only if $\mathcal{U}$ is minimal is $\mathcal{U}/\mathbb{C}^*$ compact. In this case, it is the twistor space $Z$ that fibres over $M = G/KSp(1)$.

The fundamental 3-form $\langle [X,Y], Z \rangle$ on a compact Lie algebra $g$ defines a function $f : \mathbb{G}r_3(g) \to \mathbb{R}$ for which

(i) $V \in \mathbb{G}r_3(g)$ is critical iff $V$ is a subalgebra;
(ii) $f$ achieves its max on the Wolf space of minimal $su(2)$’s;
(iii) one can easily compute $\text{Hess}(f)$ at any critical $V$.

**Example:** For $G = SU(3)$ there are two inequivalent TDS’s:

- $su(2) \subset su(3), \quad G(V) = \frac{\hat{SU}(3)}{U(2)} = \mathbb{C}P^2$
- $so(3) \subset su(3), \quad G(V) = \frac{\hat{SU}(3)}{SO(3)} = C^5$.

In the second case, $su(3)_c \cong \Sigma^2 \oplus \Sigma^4$, where $\Sigma^q = S^q(\mathbb{C}^2)$, and

$$T_{V/\mathbb{G}r_3(g)} \cong V \otimes V^\perp \cong \Sigma^2 \otimes \Sigma^4 \cong \Sigma^2 \oplus \Sigma^4 \oplus \Sigma^6$$
6.3 SU(3) continued

\[ T_\nabla \text{Gr}_3(\mathfrak{g}) \cong V \otimes V^\perp \cong \Sigma^2 \otimes \Sigma^4 \cong \Sigma^2 \oplus \Sigma^4 \oplus \Sigma^6 \]

The associated *unstable manifold* \( M^8 \) is the union of \( C^5 \) and the upward flow lines of the vector field \( \text{grad } f \). It is diffeomorphic to a rank 3 vector bundle over \( C^5 \) with fibre \( \Sigma^2 \), and

\[ T_\nu M^8 \cong \Sigma^2 \oplus \Sigma^4 \cong \Sigma^3 \otimes \Sigma^1 = \Sigma^3 \otimes H \]

is quaternionic. In fact, \( M^8 \) is locally symmetric:

\[
\frac{G_2}{SO(4)} \setminus \mathbb{C}P^2 \xrightarrow{3:1} M^8.
\]

**Theorem** For \( G \) compact simple, \( f \) is a Morse-Bott function on \( \text{Gr}_3(\mathfrak{g}) \). The unstable manifold determined by a TDS \( V \) is QK and its Swann bundle is the associated complex nilpotent orbit \( \mathcal{U} \).
7.1 Index theory

Let $M^{4n}$ be a Wolf space or a positive QK manifold with isometry group $G$. Its virtual $Spin(4n)$ representation is

$$\Delta_+ - \Delta_- = \Lambda_0^n (E - H) = \bigoplus_{p+q=n} (-1)^p R^{p,q},$$

where $R^{p,q} = \Lambda_0^p E \otimes S^q H$.

The coupled Dirac operator

$$\Gamma(M, \Delta_+ \otimes R^{p,q}) \longrightarrow \Gamma(M, \Delta_- \otimes R^{p,q})$$

has index $i^{p,q} = \int_M \text{ch}(R^{p,q}) \hat{A}(M)$. The following is an expression of Witten rigidity and a $G$-equivariant statement:

**Theorem:**

$$(-1)^p i^{p,q} = \begin{cases} 
0 & \text{if } p+q < n, \\
2p-2 + b_{2p} & \text{if } p+q = n, \\
\dim G & \text{if } p=0, q=n+2.
\end{cases}$$
7.2 Application to dimension 8

Index theory (and the $\gamma$ filtration) gives a linear constraint on the Betti numbers and estimates on the isometry group, in terms of characteristic classes including $u \in H^4(M, \mathbb{Z})$ that represents $\Omega$.

**Example:** If $\dim M = 8$ then $b_2 + 1 = b_4$, suggesting $b_2 = 0$ or 1. Moreover

$$\dim G = 5 + \int_M u^2.$$  

If $b_4 = 1$ then

$$\dim G = \begin{cases} 
5 + 16 & = \dim Sp(3), \\
5 + 9 & = \dim G_2, \\
5 + 4 & = \dim Sp(1)^3, \\
5 + 1 & = \dim SO(4), 
\end{cases}$$

corresponding to

$$\text{HP}^2 = \frac{Sp(3)}{Sp(2) \times Sp(1)}, \quad \frac{G_2}{SO(4)}, \quad \frac{\text{HP}^2}{(\mathbb{Z}_2)^2}?$$

Only the first two spaces are non-singular.
7.3 Towards a classification

Let \( M^{4n} \) be a compact positive QK manifold.

The odd Betti numbers \( b_{2p+1} \) of \( M^{4n} \) all vanish.

**Theorem:** If \( b_2(M) > 0 \) then \( M \) is isometric to the Wolf space \( \mathbb{G}_{\mathbb{R}2}(\mathbb{C}^{n+2}) \) (LeBrun-S, Wisniewski).

A proof uses Mori theory on the twistor space. If \( b_2(Z) > 1 \) there exists a second family of rational curves on \( Z \) transverse to the fibres over \( M \), and a Fano contraction \( Z \to \mathbb{C}P^{n+1} \) with its fibres tangent to the contact distribution \( D \). This forces \( Z = \mathbb{P}(T^*\mathbb{C}P^{n+1}) \).

If we ignore \( \mathbb{H}P^n \) then \( M \) is spin iff \( n \) is even, giving a dichotomy according to the parity of \( n \).
7.4 The $\hat{A}$ genus

Let $M^{4n}$ be a compact positive QK manifold. If $n$ is even, $M$ is spin and $\hat{A}(M) = 0$ because $s > 0$.

**Theorem** A positive QK manifold $M^8$ is isometric to a Wolf space (Poon-S).

An attempt to push this to dimension 12 using elliptic genera needs the assumption $\hat{A}(M) = 0$ (Herrera-Herrera).

**Theorem:** If $b_4 = 1$ and $3 \leq n \leq 6$ then $M \cong \mathbb{H}P^n$ (Amann).

All exceptional Wolf spaces have $b_4 = 1$, including $\frac{F_4}{Sp(3)Sp(1)}$.

**Theorem** If $n = 5$ and $\hat{A}(M) = 0$ then $\dim G \geq 15$ and $M$ is a Wolf space if (for example) $\int_M u^5 > 384$ (Amann).

**Question:** Does a positive QK manifold $M^{4n}$ necessarily admit an isometry group of positive dimension? Yes, at least if $n \leq 4$. If $n$ is odd, must $\hat{A}(M)$ vanish?
8.1 More topology

Consider the Poincaré polynomial

\[ P(t) = 1 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \]

of an oriented compact manifold with \( \chi = P(-1) \neq 0 \). Then

\[ \log P(t - 1) = \log \chi - dt + \phi t^2 + \cdots \]

where \( d = \dim M \), and \( 16\phi = 4P''(-1)/\chi - d^2 \). By construction, this coefficient is additive for products:

\[ \phi(M \times N) = \phi(M) + \phi(N). \]

**Theorems:**

(i) If \( M^{4n} \) is compact HK then \( \chi = 0 \) or \( \phi = -\frac{5}{6}n \).

(ii) If \( M^d = G/H \) is an irreducible compact symmetric space of type ADE or any Hermitian symmetric space, \( \phi = \frac{1}{12}(h(g) - 2)d \), where \( h(g) \) is the Coxeter number (Fino-S).

(iii) If \( M^{4n} \) is an ADE Wolf space then \( \phi = \frac{1}{3}n^2 \).
8.2 The case of $E_8$

The signature of an ADE Wolf space equals its rank: $b_{2n}^+ = b_{2n} = r$, and $\chi$ equals the number $\frac{1}{2}(\dim G - r)$ of positive roots.

$E_8/E_7Sp(1)$ has 8 primitive cohomology classes $\sigma_k \in H^{4k}(M, \mathbb{R})$;

$$H^{56}(M, \mathbb{R}) = \langle \sigma_k \cup u^{14-k} : k = 0, 3, 5, 6, 8, 9, 11, 14 \rangle,$$

exhibiting a remarkable symmetry about degree $n = 28$:

![Diagram showing cohomology classes](image)

**Question:** Is the intersection form $H^{56}(M, \mathbb{Z}) \times H^{56}(M, \mathbb{Z}) \to \mathbb{Z}$ diagonalizable or the $E_8$ lattice? (Hirzebruch-Sladowy)

The quaternionic volume (Herrera, Weingart) is:

$$\int_M u^{28} = 2^3.3^2.5^2.7.31.37.41.43.47.53 = \frac{5! \cdot 9! \cdot 57!}{19! \cdot 23! \cdot 29!} = 63468758442600$$