# Hypermultiplet moduli spaces in type II string theories: a mini-survey

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*based on work with Alexandrov and Persson, to appear, and previous work with Alexandrov, Saueressig, Vandoren, Persson, Neitzke, Gunaydin, Waldron ...*

# Propaganda

- In $D = 4$ string vacua with $N = 2$ supersymmetries , the moduli space splits into a product $\mathcal{M} = VM_4 \times HM_4$ corresponding to vector multiplets and hypermultiplets.

$$\mathrm{IIA}/\mathcal{X} \; \Big| \; \mathrm{IIB}/\hat{\mathcal{X}} \; \Big| \; \mathrm{Het}/K_3 \times T^2 \; \Big| \; \dots$$

- The study of $VM_4$ and of the BPS spectrum has had tremendous applications in mathematics and physics: classical mirror symmetry, Gromov-Witten invariants, Donaldson-Thomas invariants, black hole precision counting, etc...
- Understanding $HM_4$ may be even more rewarding: quantum extension of mirror symmetry, new geometric invariants, new checks of Het/II duality richer automorphic properties...

# Hypers = Vectors

- Upon circle compactification to $D=3$,the VM and HM moduli spaces become two sides of the same coin, exchanged by T-duality along the circle.
- $\mathrm{VM}_3$ includes $\mathrm{VM}_4$, the electric and magnetic holonomies of the $D=4$ Maxwell fields, the radius $R$ of the circle and the NUT potential $\sigma$, dual to the Kaluza-Klein gauge field in $D=3$:

$$\begin{aligned} \mathrm{VM}_3 &\approx \mathrm{c-map}(\mathrm{VM}_4) + \mathrm{1-loop} + \mathcal{O}(e^{-R}) + \mathcal{O}(e^{-R^2}) \\ \mathrm{HM}_3 &= \mathrm{HM}_4 \end{aligned}$$

- SUSY requires that both $\mathrm{VM}_3$ and $\mathrm{HM}_3$ are quaternion-Kähler manifolds.

# Instantons = Black holes + KKM

- The $\mathcal{O}(e^{-R})$ corrections come from BPS black holes in $D = 4$, whose Euclidean wordline winds around the circle: thus $VM_3$ encodes the $D = 4$ spectrum, with chemical potentials for every electric and magnetic charges, and naturally incorporates chamber dependence.

*Seiberg Witten; Shenker*

- The $\mathcal{O}(e^{-R^2})$ corrections come from Kaluza-Klein monopoles, i.e. gravitational instantons of the form $\mathrm{TN}_k \times \mathcal{Y}$ ($\mathcal{Y} = \hat{\mathcal{X}}, \mathcal{X}, K_3 \times T^2$). (in Lorentzian signature, these would have closed timelike curves).
- Including these additional contributions will (hopefully) lead to enhanced automorphic properties, analogous to the $SL(2,\mathbb{Z}) \to Sp(2,\mathbb{Z})$ enhancement in $N = 4$ dyon counting.

*Dijkgraaf Verlinde Verlinde; Gunaydin Neitzke BP Waldron*

# SYM vs. SUGRA

- A much simpler version of this problem occurs in (Seiberg-Witten) $\mathcal{N} = 2$ SYM field theories on $\mathbb{R}^3 \times S^1$. In this case $VM_3$ is a hyperkähler manifold of the form

$$\mathrm{VM}_3 \approx \mathrm{rigid\ c-map}(\mathrm{VM}_4) + \mathcal{O}(e^{-R})$$

- The $\mathcal{O}(e^{-R})$ corrections similarly come from BPS dyons in $D = 4$. Understanding their effect on the complex symplectic structure of the twistor space $\mathcal{Z}$ of $\mathrm{VM}_3$ has lead to a physical derivation of the KS wall-crossing formula.

*Gaiotto Moore Neitzke, Kontsevich Soibelman*

- The extension to $\mathcal{N} = 2$ SUGRA is non-trivial, due (in part) to the exponential growth of BPS degeneracies, and lack of a good description of KK monopoles. In fact, KKM contributions appears to needed in order to resolve the ambiguity of the black hole asymptotic series.

*BP Vandoren*

# Back to HM

- On the flip side of the coin, $R = 1/g_{(4)}$ is the inverse string coupling. The $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections to $\mathrm{HM}_4$ now originate from Euclidean D-branes and NS5-branes, respectively.

*Becker Becker Strominger*

- When $\mathcal{X}$ is K3-fibered, $HM_4$ can in principle be computed exactly using Het/type II duality: since the heterotic string coupling belongs to $\mathrm{VM}_4$, $HM_4$ is determined by the $(0,4)$ heterotic SCFT at tree level (still non-trivial due to non-perturbative $\alpha'$ corrections)

*Aspinwall*

- Recent progress has instead occurred on the type II side, combining S-duality and mirror symmetry with an improved understanding of twistor techniques.

*Robles-Llana Rocek Saueressig Theis Vandoren*

*Alexandrov BP Saueressig Vandoren*

# Outline

# Outline

# The perturbative metric I

- The HM moduli space in type IIA compactified on a CY 3-fold (family) $\mathcal{X}$ is a quaternion-Kähler manifold $\mathcal{M}$ of real dimension $2b_3(\mathcal{X}) = 4(h_{2,1}+1)$.
- $\mathcal{M} \equiv \mathcal{Q}_c(\mathcal{X})$ encodes
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complex structure of the CY family $\mathcal{X}$,
  3. the periods of the RR 3-form $C$ on $\mathcal{X}$,
  4. the NS axion $\sigma$, dual to the Kalb-Ramond $B$-field in 4D
- To write down the metric explicitly, let us choose a symplectic basis $\mathcal{A}^\Lambda, \mathcal{B}_\Lambda$, $\Lambda = 0 \dots h_{2,1}$ of $H_3(\mathcal{X}, \mathbb{Z})$.

# The perturbative metric II

- The complex structure moduli space $\mathcal{M}_c(\mathcal{X})$ may be parametrized by the periods $\Omega(z^a) = (X^\Lambda, F_\Lambda) \in H_3(\mathcal{X}, \mathbb{C})$ of the (3,0) form

$$X^\Lambda = \int_{\mathcal{A}^\Lambda} \Omega_{3,0}\,, \quad F_\Lambda = \int_{\mathcal{B}_\Lambda} \Omega_{3,0}\,,$$

  up to holomorphic rescalings $\Omega \mapsto e^f \Omega$.

- $\mathcal{M}_c(\mathcal{X})$ is endowed with a special Kähler metric

$$\mathrm{d}s^2_{\mathcal{SK}} = \partial\bar\partial\mathcal{K}\,, \qquad \mathcal{K} = -\log[\mathrm{i}(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)]$$

  and a $\mathbb{C}^\times$ bundle $\mathcal{L}$ with connection $\mathcal{A}_K = \frac{\mathrm{i}}{2}(\mathcal{K}_a \mathrm{d}z^a - \mathcal{K}_{\bar a}\mathrm{d}\bar{z}^{\bar a})$.

- $\Omega$ transforms as $\Omega \mapsto e^f \rho(M)\,\Omega$ under a monodromy $M$ in $\mathcal{M}_c(\mathcal{X})$, where $\rho(M) \in Sp(b_3, \mathbb{Z})$.

# The perturbative metric III

- Topologically trivial harmonic C-fields on $\mathcal{X}$ may be parametrized by the real periods $C = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C, \quad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C\,.$$

- Large gauge transformations require that $C$ lives in the intermediate Jacobian torus

$$C \in T = H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z})$$

  i.e. that $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ have unit periodicities.
- This is consistent with D-instanton charge quantization, as we shall discuss later.

# The perturbative metric IV

- $T$ carries a canonical symplectic form and complex structure induced by the Hodge $\star_{\mathcal{X}}$, hence a Kähler metric

$$\mathrm{d}s_T^2 = -\frac{1}{2}(\mathrm{d}\tilde{\zeta}_\Lambda - \bar{\mathcal{N}}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'})\mathrm{Im}\mathcal{N}^{\Lambda\Sigma}(\mathrm{d}\tilde{\zeta}_\Lambda - \mathcal{N}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'})$$

where $\mathcal{N}$ is the (Weil) period matrix ($\mathrm{Im}\mathcal{N} < 0$),

$$\mathcal{N}_{\Lambda\Lambda'} = \bar{\tau}_{\Lambda\Lambda'} + 2\mathrm{i}\frac{[\mathrm{Im}\tau\cdot X]_\Lambda[\mathrm{Im}\tau\cdot X]_{\Lambda'}}{X^\Sigma\,\mathrm{Im}\tau_{\Sigma\Sigma'}X^{\Sigma'}}\,.$$

while $\tau_{\Lambda\Sigma} = \partial_{X^\Lambda}\partial_{X^\Sigma}F$ is the Griffiths period matrix.

- Under monodromies, $C \mapsto \rho(M)C$ . We shall refer to the total space of the torus bundle $T \to \mathcal{J}_c(\mathcal{X}) \to \mathcal{M}_c(\mathcal{X})$ as the (Weil) intermediate Jacobian of $\mathcal{X}$.

*see also Stienstra's talk*

# The tree-level metric

- At tree level, i.e. in the strict weak coupling limit $R = \infty$, the quaternion-Kähler metric on $\mathcal{M}$ is given by the $c$-map metric

*Cecotti Girardello Ferrara; Ferrara Sabharwal*

$$ds^2_{\mathcal{M}} = \frac{4}{R^2}\,\mathrm{d}R^2 + 4\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{1}{16R^4} D\sigma^2 \,.$$

where

$$D\sigma \equiv \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle = \mathrm{d}\sigma + \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda$$

- The $c$-map metric admits continuous isometries

$$T_{H,\kappa} : (C,\sigma) \mapsto (C + H, \sigma + \kappa + \langle C, H\rangle)$$

where $H \in H^3(\mathcal{X},\mathbb{R})$ and $\kappa \in \mathbb{R}$, satisfying the Heisenberg group relation

$$T_{H_1,\kappa_1}\, T_{H_2,\kappa_2} = T_{H_1+H_2,\kappa_1+\kappa_2+\langle H_1,H_2\rangle} \,.$$

# The one-loop corrected metric I

- The one-loop correction deforms the metric on $\mathcal{M}$ into

$$ds^2_{\mathcal{M}} = 4\frac{R^2+2c}{R^2(R^2+c)}\,\mathrm{d}R^2 + \frac{4(R^2+c)}{R^2}\,\mathrm{d}s^2_{\mathcal{SK}} + \frac{\mathrm{d}s^2_T}{R^2} + \frac{2\,c}{R^4}\,e^{\mathcal{K}}\,|X^\Lambda \mathrm{d}\tilde\zeta_\Lambda - F_\Lambda \mathrm{d}\zeta^\Lambda|^2 + \frac{R^2+c}{16R^4(R^2+2c)}D\sigma^2\,.$$

where $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle + 8c\mathcal{A}_K\,, \quad c = -\frac{\chi(\mathcal{X})}{192\pi}$

*Antoniadis Minasian Theisen Vanhove; Gunther Hermann Louis; Robles-Llana Saueressig Vandoren*

- The one-loop correction to $g_{rr}$ was computed by reducing the CP-even $R^4$ coupling in 10D on $\mathcal{X}$. The correction to $D\sigma$ can be obtained with less effort by reducing CP-odd couplings in 10D.

# The one-loop corrected metric II

- Consider the topological coupling in $D = 10$ type IIA supergravity:

$$\int_{\mathcal{Y}} \left( \frac{1}{6} B \wedge \mathrm{d}B \wedge \mathrm{d}B - B \wedge I_8 \right) , \quad I_8 = \frac{1}{48}(p_2 - \frac{1}{4} p_1^2)$$

- On a complex 10-manifold,

$$B \wedge I_8 = \frac{1}{24} B \wedge \left[ c_4 - c_1 \left( c_3 + \frac{1}{8} c_1^3 - \frac{1}{2} c_1 c_2 \right) \right] .$$

- Integrating on $\mathcal{X}$ and using $c_4 = 0, c_3 = \chi(\mathcal{X}), c_1 = -\omega_c$ leads to

$$\int d^4 x \left[ \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} (\mathrm{d}C^\Lambda + \zeta^\Lambda \mathrm{d}B) \wedge d\zeta^\Sigma - \frac{\chi(\mathcal{X})}{24\pi} B \wedge \omega_c \right]$$

where $C^\Lambda = \int_{\mathcal{A}^\Lambda} C$. Dualizing the two-forms $C^\Lambda, B$ into $\tilde{\zeta}_\Lambda, \sigma$ produces the one-form $D\sigma$ indicated previously.

# The one-loop corrected metric III

- The one-loop correction to $D\sigma$ has important implications for the topology of the HM moduli space, as we shall discuss later.
- The one-loop corrected metric is presumably exact to all orders in $1/R$. It will receive $\mathcal{O}(e^{-R})$ and $\mathcal{O}(e^{-R^2})$ corrections from D-instantons and NS5-brane instantons, eventually breaking all continuous isometries.
- Note the curvature singularity at finite distance $R^2 = -2c$ when $\chi(\mathcal{X}) > 0$ ! This should hopefully be resolved by instanton corrections.

# A lightning review of twistors I

- QK manifolds $\mathcal{M}$ are conveniently described via their twistor space $\mathbb{P}^1 \to \mathcal{Z} \to \mathcal{M}$, a complex manifold equipped with a canonical complex contact structure. Choosing a stereographic coordinate $t$ on $\mathbb{P}^1$, the contact structure is the kernel of the local (1,0)-form

$$Dt = \mathrm{d}t + p_+ - \mathrm{i}p_3 t + p_- t^2$$

where $p_3, p_\pm$ are the $SU(2)$ components of the Levi-Civita connection on $\mathcal{M}$.

- $\mathcal{Z}$ is further equipped with a Kähler-Einstein metric

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|Dt|^2}{(1+t\bar{t})^2} + \frac{\nu}{4}\mathrm{d}s^2_{\mathcal{M}}\ , \qquad \nu = \frac{R(\mathcal{M})}{4d(d+2)}$$

If $\mathcal{M}$ has negative scalar curvature, $\mathcal{Z}$ is pseudo-Kähler with signature $(2, \dim \mathcal{M})$.

*see Salamon's lecture*

# A lightning review of twistors II

- Rk: in case one is not willing to work with contact geometry, one may equivalently consider the HK cone, a $\mathbb{C}^\times$ bundle over $\mathcal{Z}$, which carries instead a homogeneous complex symplectic structure.

*Swann; de Wit Rocek Vandoren*

- Locally, there always exist Darboux coordinates $\Xi = (\xi^\Lambda, \rho_\Lambda)$ and $\tilde\alpha$ such that

$$Dt \propto \mathrm{d}\tilde\alpha + \xi^\Lambda \mathrm{d}\rho_\Lambda - \rho_\Lambda \mathrm{d}\xi^\Lambda = \mathrm{d}\tilde\alpha + \langle \Xi, \mathrm{d}\Xi \rangle \ .$$

*Alexandrov BP Saueressig Vandoren*

# A lightning review of twistors III

- By the moment map construction, continuous isometries of $\mathcal{M}$ are in 1-1 correspondence with classes in $H^0(\mathcal{Z}, \mathcal{O}(2))$. In particular, any continuous isometry of $\mathcal{M}$ can be lifted to a holomorphic action on $\mathcal{Z}$.

*Salamon; Galicki Salamon*

- Infinitesimal deformations of $\mathcal{M}$ lift to deformations of the complex contact transformations between Darboux coordinate patches on $\mathcal{Z}$, hence are classified by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

*Lebrun; Alexandrov BP Saueressig Vandoren*

# Twistor description of the perturbative metric

- For the one-loop corrected HM metric, the following Darboux coordinates do the job, away from the north and south poles $t = 0, \infty$:

$$\Xi = C + 2(R^2 + c)\, e^{\mathcal{K}/2} \left[ t^{-1}\Omega - t\,\bar{\Omega} \right]$$

$$\tilde{\alpha} = \sigma + 2(R^2 + c)\, e^{\mathcal{K}/2} \left[ t^{-1}\langle \Omega, C \rangle - t\,\langle \bar{\Omega}, C \rangle \right] - 8\mathrm{i}c \log t$$

*Neitzke BP Vandoren; Alexandrov*

- The isometry $T_{H,\kappa}$ acts holomorphically on $\mathcal{Z}$ by

$$(\Xi, \tilde{\alpha}) \mapsto (\Xi + H, \tilde{\alpha} + 2\kappa + \langle \Xi, H \rangle)$$

# Outline

# Topology of the HM moduli space I

- At least at weak coupling, $\mathcal{M}$ is foliated by hypersurfaces $\mathcal{C}(R)$ of constant string coupling. We shall now discuss the topology of the leaves $\mathcal{C}(R)$, which is independent of $R$.
- Quotienting by translations along the NS axion $\sigma$, $\mathcal{C}/\partial_\sigma$ reduces to the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$, in particular $C \in T = H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z})$.
- This is consistent with the fact that Euclidean D2-branes wrapping a special Lagrangian submanifold in integer homology class $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in H_3(\mathcal{X}, \mathbb{Z})$ induce corrections of the form

$$\delta ds^2|_{D2} \sim \exp\left( -8\pi \frac{|Z_\gamma|}{g_{(4)}} - 2\pi i \langle \gamma, C \rangle \right) ,$$

where $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the central charge.

# Topology of the HM moduli space II

- Continuous translations along $\sigma$ will be broken by NS5-brane instantons to discrete shifts $\sigma \mapsto \sigma + 2$ (in our conventions). Thus $e^{\mathrm{i}\pi\sigma}$ parametrizes the fiber of a circle bundle $\mathcal{C}$ over $\mathcal{J}_c(\mathcal{X})$, to be determined.
- The horizontal one-form $D\sigma = \mathrm{d}\sigma + \langle C, \mathrm{d}C\rangle - \frac{\chi(\mathcal{X})}{24\pi}\mathcal{A}_K$ implies that

$$c_1(\mathcal{C}) = \mathrm{d}\left(\frac{D\sigma}{2}\right) = \omega_T + \frac{\chi(\mathcal{X})}{24}\omega_c$$

where $\omega_T = \mathrm{d}\tilde{\zeta}_\Lambda \wedge \mathrm{d}\zeta^\Lambda, \omega_c = -\frac{1}{2\pi}\mathrm{d}\mathcal{A}_K$ are the Kähler forms on $T$ and $\mathcal{M}_c(\mathcal{X})$, respectively.

# Five-brane instantons I

- NS5-brane instantons with charge $k \in \mathbb{Z}$ are expected to produce corrections to the metric of the form

$$\delta \mathrm{d}s^2|_{\mathrm{NS5}} \sim \exp\left(-4\pi|k|/g_{(4)}^2 - \mathrm{i}k\pi\sigma\right) \mathcal{Z}^{(k)}(z^a, C) ,$$

  where $\mathcal{Z}^{(k)} = \mathrm{Tr}(F^2(-1)^F)$ is the (twisted) partition function of the world-volume theory on a stack $k$ five-branes. For this to be globally well-defined, $\mathcal{Z}^{(k)}$ must be a section of $\mathcal{C}^k$.

- Recall that the type IIA NS5-brane supports a self-dual 3-form flux $H = \mathrm{i} \star H$, together with its SUSY partners. The partition function of a self-dual form is known to be a holomorphic section of a non-trivial line bundle $\mathcal{L}_{\mathrm{NS5}}^k$ over the space of metrics and $C$ fields.

*Witten; Henningson Nilsson Salomonson; Belov Moore; ...*

# Five-brane instantons II

- Indeed, the restriction $\mathcal{L}_{\mathrm{NS5}}|_T$ is known to be a line bundle with first Chern class $c_1 = \omega_T$. To specify this bundle, one must choose holonomies $\sigma(H) \in U(1)$ around each cycle $H \in H_3(\mathcal{X}, \mathbb{Z})$, such that

$$\sigma(H + H') = (-1)^{\langle H, H' \rangle} \, \sigma(H) \, \sigma(H') \, .$$

  Thus, $\sigma(H)$ defines a quadratic refinement of the intersection form on $H^3(\mathcal{X}, \mathbb{Z})$. *Do not confuse* $\sigma(H)$ *with the NS-axion* $\sigma$ *!*

- The general solution can be parametrized by characteristics $\Theta \in H_3(\mathcal{X}, \mathbb{R}) / H_3(\mathcal{X}, \mathbb{Z})$ (notation: $\mathrm{E}^x \equiv e^{2\pi \mathrm{i} x}$)

$$\sigma(H) = \mathrm{E}^{-\frac{1}{2} n^\Lambda m_\Lambda + \langle H, \Theta \rangle} \, , \quad H = (n^\Lambda, m_\Lambda)$$

# Five-brane instantons III

- Rk: $\sigma(H)$ need not be $\pm 1$, and $\Theta$ may depend on the metric of $\mathcal{X}$. It can be computed in principle from M-theory.

*Diaconescu Moore Witten*

- The same quadratic refinement also appears in the normalization factor $\Omega(\gamma)\sigma(\gamma)$ of D-instantons corrections, for consistency with the KS formula. It would be interesting to derive it from a one-loop determinant.

*Gaiotto Moore Neitzke; Harvey Moore*

- The bundle $(\mathcal{L}_\Theta)^k$ is then defined by the twisted periodicity condition

$$\mathcal{Z}(\mathcal{N}, C+H) = \sigma_\Theta^k(H)\, \mathrm{E}^{\frac{k}{2}\langle H,C\rangle} \mathcal{Z}(\mathcal{N}, C)$$

# Five-brane instantons IV

- At weak coupling, the partition function of a chiral five-brane can be obtained by holomorphic factorization of the partition function of a non-chiral 3-form $H = \mathrm{d}\mathcal{B}$ on $\mathcal{X}$, with Gaussian action. This leads to a Siegel theta series of rank $b_3(\mathcal{X})$, level $k/2$ satisfying the above periodicity property:

$$\mathcal{Z}_\mu^{(k)}(\mathcal{N}, C) = N \sum_{n \in \Gamma_m + \mu + \theta} \mathrm{E}^{\frac{k}{2}(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde{\zeta}_\Lambda - \phi_\Lambda)n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)},$$

where $\Gamma_m$ is a Lagrangian sublattice of $\Gamma = H^3(\mathcal{X}, \mathbb{Z})$, $N$ is a $C$-independent normalization factor, and $\mu$ runs over $(\Gamma_m/k)/\Gamma_m$, i.e. over the $|k|^{b_3/2}$ independent holomorphic sections of $\mathcal{L}_\Theta^k$.

# Topology of the NS axion I

- For the coupling $e^{-\mathrm{i}\pi k\sigma}\mathcal{Z}^{(k)}$ to be invariant under large gauge transformations, $e^{\mathrm{i}\pi\sigma}$ must also transform as a section of $\mathcal{L}_\Theta$. Therefore, $\sigma$ must pick up additional shifts under discrete translations along $T$,

$$T'_{H,\kappa} : (C,\sigma) \mapsto \left(C+H, \sigma+\kappa+\langle C,H\rangle - \frac{1}{2} n^\Lambda m_\Lambda + \langle H,\Theta\rangle\right)$$

where $H \equiv (n^\Lambda, m_\Lambda) \in \mathbb{Z}^{b_3}$, $p \in \mathbb{Z}$.

- As a result, the large gauge transformations now form an Abelian group,

$$T'_{H_1,\kappa_1} T'_{H_2,\kappa_2} = T'_{H_1+H_2,\kappa_1+\kappa_2}.$$

# Topology of the NS axion II

- The second term in $c_1(\mathcal{C}) = \omega_T + \frac{\chi_{\mathcal{X}}}{24}\omega_c$ implies that $e^{\mathrm{i}\pi\sigma}$ in addition transforms as a section of $\mathcal{L}^{\chi_{\mathcal{X}}/24}$ under monodromies.
- Thus, $\mathcal{M}$ is (at least at weak coupling) foliated by hypersurfaces $\mathcal{C}$ which are topologically a circle bundle $\mathcal{L}^{-\chi(\mathcal{X})/24} \otimes \mathcal{L}_{\Theta}$ over the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$.
- Using insights from topological strings, one finds that the suitably normalized $\mathcal{Z}^{(k)}$ transforms as a section of $\mathcal{L}^{\chi_{\mathcal{X}}/24-2} \otimes \mathcal{L}_{\Theta}$, hence $e^{-\mathrm{i}\pi k\sigma}\mathcal{Z}^{(k)}$ is not monodromy invariant. Daniel will discuss how to resolve this discrepancy in his talk.
- By mirror symmetry, the type IIB HM moduli space is similarly foliated by circle bundles over the symplectic Jacobian $\mathcal{J}_K(\hat{\mathcal{X}})$, as we now discuss.

# Outline

- The HM moduli space in type IIB compactified on a CY 3-fold $\hat{\mathcal{X}}$ is a QK manifold $\mathcal{M} \equiv \mathcal{Q}_K(\hat{\mathcal{X}})$ of real dimension $4(h_{1,1}+1)$
  1. the 4D dilaton $R \equiv 1/g_{(4)}$,
  2. the complexified Kähler moduli $z^a = b^a + \mathrm{i} t^a = X^a/X^0$
  3. the periods of $C = C^{(0)} + C^{(2)} + C^{(4)} + C^{(6)} \in H^{\mathrm{even}}(\hat{\mathcal{X}}, \mathbb{R})$
  4. the NS axion $\sigma$
- Near the infinite volume point, $\mathcal{M}_K(\hat{\mathcal{X}})$ is governed by

$$F(X) = -\frac{N(X^a)}{X^0} + \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma + \chi(\hat{\mathcal{X}}) \frac{\zeta(3)(X^0)^2}{2(2\pi \mathrm{i})^3} + F_{\mathrm{GW}}(X)$$

where $N(X^a) \equiv \frac{1}{6} \kappa_{abc} X^a X^b X^c$, $\kappa_{abc}$ is the cubic intersection form, $A_{\Lambda\Sigma}$ is a constant, real symmetric matrix, defined up to integer shifts and $F_{\mathrm{GW}}$ are Gromov-Witten instanton corrections.

# HM moduli space in type IIB II

- Quantum mirror symmetry implies $\mathcal{Q}_c(\mathcal{X}) = \mathcal{Q}_K(\hat{\mathcal{X}})$. At the perturbative level, this reduces to classical mirror symmetry.
- D-instantons are now Euclidean D5-D3-D1-D(-1), described mathematically by coherent sheaves $E$ on $\mathcal{X}$. Their charge vector $\gamma$ is related to the Chern classes via the Mukai map

$$q_\Lambda X^\Lambda - p^\Lambda F_\Lambda = \int_{\hat{\mathcal{X}}} e^{-(B+\mathrm{i}J)} \,\mathrm{ch}(E) \sqrt{\mathrm{Td}(\hat{\mathcal{X}})}$$

- Assuming that $A_{\Lambda\Sigma}$ satisfies the congruences

$$A_{00} \in \mathbb{Z}\,, \quad A_{0a} \in \frac{c_{2,a}}{24} + \mathbb{Z}\,, \quad \frac{1}{2}\kappa_{abc} p^b p^c - A_{ab} p^b \in \mathbb{Z} \quad \text{for } \forall p^a \in \mathbb{Z}\,,$$

  the D-instanton charge vector $\gamma \in H^{\rm even}(\hat{\mathcal{X}}, \mathbb{Z})$, hence $C$ takes values in the symplectic Jacobian $T = H^{\rm even}(\hat{\mathcal{X}}, \mathbb{R})/H^{\rm even}(\hat{\mathcal{X}}, \mathbb{Z})$.

# HM moduli space in type IIB III

- It is often convenient to eliminate $A_{\Lambda\Sigma}$ by a non-integer symplectic transformation, leading to non-integer electric charges $q'_\Lambda$,

$$q'_\Lambda = q_\Lambda - A_{\Lambda\Sigma} p^\Sigma \, , \quad \tilde\zeta'_\Lambda = \tilde\zeta_\Lambda - A_{\Lambda\Sigma}\zeta^\Lambda \, , \quad F' = F - \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma$$

$$q'_a \in \mathbb{Z} - \frac{p^0}{24} c_{2,a} - \frac{1}{2}\kappa_{abc} p^b p^c , \qquad q'_0 \in \mathbb{Z} - \frac{1}{24} p^a c_{2,a} \, ,$$

- The HM metric should admit an isometric action of $SL(2,\mathbb{Z})$, corresponding to type IIB S-duality in 10 dimensions. This action is most easily described in the “primed" frame.

# S-duality in twistor space I

- An element $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ acts holomorphically on $\mathcal{Z}$ via

$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d}, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d},$$

$$\rho'_a \mapsto \rho'_a + \frac{c}{2(c\xi^0 + d)} \kappa_{abc}\xi^b\xi^c - c_{2,a}\,\varepsilon(\delta),$$

$$\begin{pmatrix} \frac{\mathrm{i}}{2}\rho'_0 \\ \alpha' \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \frac{\mathrm{i}}{2}\rho'_0 \\ \alpha' \end{pmatrix} + \dots$$

where $\alpha' = (\tilde\alpha + \xi^\Lambda \rho'_\Lambda)/(4\mathrm{i})$, and $\epsilon(\delta)$ is the multiplier system of the Dedekind eta function,

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right)/\eta(\tau) = \mathrm{E}^{\epsilon(\delta)}(c\tau + d)^{1/2}.$$

# S-duality in twistor space II

- The transformation rule of $\rho'_a$ can be summarized by saying that $\mathrm{E}^{p^a \rho'_a}$ transforms like the automorphy factor of a multi-variable Jacobi form of index $m_{ab} = \frac{1}{2}\kappa_{abc} p^c$ and multiplier system $\mathrm{E}^{-c_{2a} p^a \epsilon(\delta)}$.
- This is consistent with the multiplier system $\mathrm{E}^{-c_{2a} p^a \epsilon(\delta)}$ of the D4-D2-D0 partition function, which should describe D-instanton corrections to $VM_3$ with vanishing D6-brane charge in type IIA/$\mathcal{X}$.

*Denef Moore; Manschot*

# S-duality in twistor space III

- S-duality relates D5 and NS5. Starting from the known form of D5-D3-D1-D(-1) corrections, one may construct a Poincare series to obtain the contributions from $k$ five branes. This leads to a non-Gaussian generalization of the Siegel theta series, closely related to the topological string amplitude.

  *see Persson's talk*

- If indeed monodromy invariance, Heisenberg invariance and S-duality hold simultaneously, the exact HM moduli space will exhibit enhanced automorphic properties, as we now discuss.

# Automorphy

## Symmetries

- Monodromies

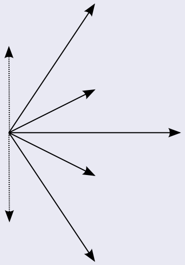
## Root diagram (2D projection)

# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.

## Root diagram (2D projection)

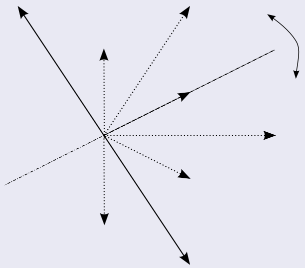


# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.
- S-duality

## Root diagram (2D projection)

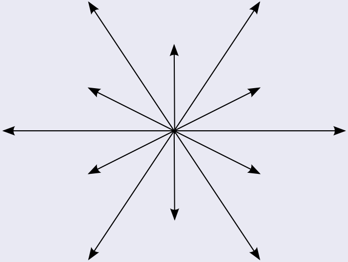


# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.

## Root diagram (2D projection)



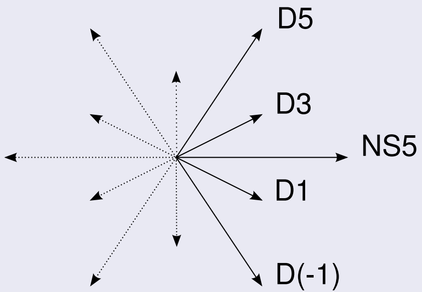
*Alexeevsky; Gunaydin Koepsell Nicolai;Gunaydin Neitzke Pavlyk BP*

# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 3-step nilpotent

## Root diagram (2D projection)

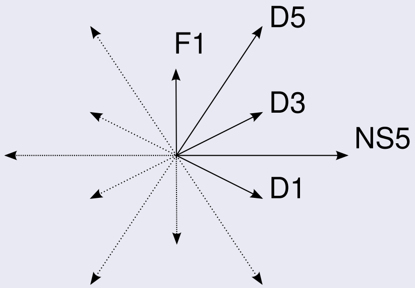


# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 3-step nilpotent
- 5-step nilpotent

## Root diagram (2D projection)

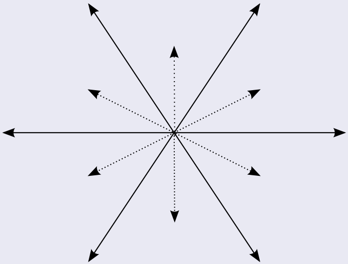


# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 3-step nilpotent
- 5-step nilpotent
- Long roots: SL(3)

## Root diagram (2D projection)



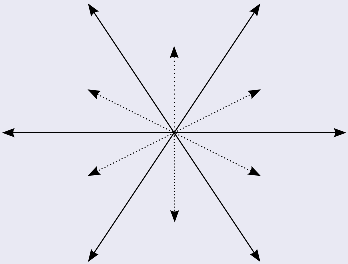
*Persson BP*

# Automorphy

## Symmetries

- Monodromies
- Large gauge transf.
- S-duality
- Quasiconformal sym.
- 3-step nilpotent
- 5-step nilpotent
- Long roots: SL(3)
- Rigid case: SU(2,1)

## Root diagram (2D projection)



*Bao Kleinschmidt Nilsson Persson BP*

# Outline

# Conclusion I

- We have determined the topology of the HM moduli space in type IIA$/\mathcal{X}$ at fixed (weak) coupling:

$$\mathcal{M} = \mathbb{R}^+_r \times \begin{pmatrix} S^1_\sigma & \rightarrow & \mathcal{C}(r) \\ & & \downarrow \\ & & \mathcal{J}(\mathcal{X}) \end{pmatrix} ,$$

  where $\mathcal{J}_c(\mathcal{X})$ is the intermediate Jacobian of the CY family $\mathcal{X}$, $\mathcal{C}(r)$ is the circle bundle $\mathcal{L}_\Theta \otimes \mathcal{L}^{\chi(\mathcal{X})/24}$. It would be very interesting to compute the characteristics $\Theta$ from M-theory.
- The same holds in type IIB$/\hat{\mathcal{X}}$ by replacing $\mathcal{J}_c$ by the symplectic Jacobian, i.e. the torus bundle over $\mathcal{M}_K(\hat{\mathcal{X}})$ with fiber $H^{\rm even}(\hat{\mathcal{X}}, \mathbb{R})/K(\hat{\mathcal{X}})$, and $\chi(\mathcal{X}) \rightarrow -\chi_{\hat{\mathcal{X}}}$.

# Conclusion II

- D-instanton corrections are essentially under control, barring the important issue of the divergence of the D-instanton series. We have taken some steps in understanding five-brane corrections.
- What is the topology of the full HM space ? Is the singularity at $R^2 = \chi(\mathcal{X})/96\pi$ resolved by quantum effects ?
- Physical reasoning suggests that $\mathcal{M}$ admits a large arithmetic group of isometries. Can this be used to compute generalized or even motivic DT invariants ?
- We seem to be missing an efficient way of describing discrete isometries directly in twistor space...