Fivebrane Instantons and Hypermultiplets

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Based on [Alexandrov, D.P., Pioline] to appear

Also related: [Pioline, D.P., 0902.3274]
[Bao, Kleinschmidt, Nilsson, D.P., Pioline, 0909.4299 & 1005.4848]
So far: **perturbative** and **D-instanton** corrections to the hypermultiplet metric

This talk: take initial steps towards completing the picture

[Becker, Becker, Strominger]
include NS5-branes

IIA

- worldvolume theory is chiral
  - partition function is a section of a line bundle over the space of 3-forms
  - requires a choice of "quadratic refinement"

IIB

- worldvolume theory is non-chiral
  - related to D5 by S-duality
  - DT-invariants and topological strings

non-trivial contact structure on twistor space \( \mathcal{Z}_\mathcal{M} \rightarrow \mathcal{M} \)

How to implement NS5-corrections to \( \mathcal{M} \)?
Outline

- The HM moduli space in IIA/CY3
- The type IIA fivebrane partition function revisited
- Uplifting to twistor space
- NS5-brane instantons from S-duality
- Conclusions and Discussion
Type IIA point of view on $\mathcal{M}$  (recap from Boris’s talk)

In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian

$$\mathcal{M} \sim \mathbb{R}_\phi^+ \times \left( \frac{S^1_\sigma}{\mathcal{C}(\phi)} \rightarrow \mathcal{J}_c(\mathcal{X}) \right)$$

$$g_s = e^{-\phi/2}$$

$d\text{im}\mathcal{M} = 4(h_{2,1} + 1)$

- Dual of $B_{(2)}$ in $D = 4$
- “NS-axion”
Type IIA point of view on $\mathcal{M}$  

(recap from Boris’s talk)

In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian

$$\mathcal{M} \sim \mathbb{R}^+_\phi \times \begin{pmatrix} S^{1}_\sigma \rightarrow C(\phi) \\ \downarrow \mathcal{J}_c(\mathcal{X}) \end{pmatrix}$$

$$\dim \mathcal{M} = 4(h_{2,1} + 1)$$

$$H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z}) \rightarrow \mathcal{J}_c(\mathcal{X}) \downarrow \mathcal{M}_c(\mathcal{X}),$$

$$\zeta^\Lambda = \int_{A^\Lambda} C_{(3)} \quad \tilde{\zeta}_\Lambda = \int_{B^\Lambda} C_{(3)}$$

$$(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in \frac{H^3(\mathcal{X}, \mathbb{R})}{H^3(\mathcal{X}, \mathbb{Z})} = \mathcal{T}$$

$$\dim \mathcal{T} = 2h_{2,1} + 2$$
Type IIA point of view on $\mathcal{M}$ (recap from Boris’s talk)

In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian

\[
\mathcal{M} \sim \mathbb{R}_\phi^+ \times \left( \frac{S_\sigma^1}{C(\phi)} \right) \quad \text{dim } \mathcal{M} = 4(h_{2,1} + 1)
\]

\[
H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z}) \rightarrow \mathcal{J}_c(\mathcal{X})
\]

\[
\mathcal{M}_c(\mathcal{X}), \quad \text{dim } \mathcal{M}_c = 2h_{2,1}
\]

\[
X^I = \int_{\mathcal{A}^I} \Omega
\]

\[
(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in \frac{H^3(\mathcal{X}, \mathbb{R})}{H^3(\mathcal{X}, \mathbb{Z})} = \mathcal{T}
\]

\[
\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C_{(3)} \quad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}^\Lambda} C_{(3)}
\]
In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian:

\[ \mathcal{M} \sim \mathbb{R}^+_{\phi} \times \left( \begin{array}{c} S^1_{\sigma} \\ \downarrow \\ \mathcal{J}_c(\mathcal{X}) \end{array} \right) \]

First Chern class:

\[ c_1(\mathcal{C}) = d \left( \frac{D\sigma}{2} \right) = \omega_T + \frac{\chi}{24} \omega_{\mathcal{M}_c} \]

- classical contribution
- one-loop correction

\( \chi \) Euler number of \( \mathcal{X} \)
Type IIA point of view on $\mathcal{M}$  

(Recap from Boris’s talk)

In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian

\[
\mathcal{M} \sim \mathbb{R}^+_{\phi} \times \left( \begin{array}{c}
S^1_{\sigma} \\ \mathcal{C}(\phi) \\ \mathcal{J}_c(\mathcal{X})
\end{array} \right)
\]

Translations along $\mathcal{T}$ form a Heisenberg group

- $\zeta^\Lambda \longleftrightarrow \zeta^\Lambda + n^\Lambda$
- $\tilde{\zeta}^\Lambda \longleftrightarrow \tilde{\zeta}^\Lambda + m^\Lambda$
- $\sigma \longleftrightarrow \sigma + 2\kappa \rightarrow m^\Lambda \zeta^\Lambda + n^\Lambda \tilde{\zeta}^\Lambda$

Broken by D2-instantons

\[n^\Lambda, m^\Lambda \in \mathbb{Z}\]

Broken by NS5-instantons

\[\kappa \in \mathbb{Z}\]
Type IIA point of view on $\mathcal{M}$ (recap from Boris’s talk)

In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian

$$\mathcal{M} \sim \mathbb{R}^+_\phi \times \begin{pmatrix} \frac{S^1_\sigma}{\sigma} & \rightarrow & \mathcal{C}(\phi) \\ \downarrow & & \downarrow \\ \mathcal{J}_c(\mathcal{X}) \end{pmatrix}$$

Translations along $\mathcal{T}$ form a Heisenberg group

- $\zeta^\Lambda \mapsto \zeta^\Lambda + n^\Lambda$
- $\tilde{\zeta}_\Lambda \mapsto \tilde{\zeta}_\Lambda + m_\Lambda$
- $\sigma \mapsto \sigma + 2\kappa - m_\Lambda \zeta^\Lambda + n^\Lambda \tilde{\zeta}_\Lambda + c_\Theta(m, n)$

**cocycle**

required for consistency of NS5-instantons
Qualitative form of NS5-instanton corrections

\[ ds^2_M \big|_{\text{NS}_5} \sim e^{-4\pi |k|} e^{\phi - i\pi k \sigma} Z(k)(\zeta, \tilde{\zeta}) \]

chiral NS5-partition function
Qualitative form of NS5-instanton corrections

\[ ds^2_{\mathcal{M}} \mid_{\text{NS5}} \sim e^{-4\pi|k|} e^{\phi - i\pi k\sigma} \mathcal{Z}^{(k)}(\zeta, \tilde{\zeta}) \]

- NS5-instanton action: \[ S_{\text{NS5}} = 4\pi|k| \left( g_s^{-2} + \cdots \right) + i\pi \left( \sigma + \cdots \right) \]
- \( e^{-i\pi k\sigma} \) is valued in the circle bundle \( \mathcal{C}^{-k} \)
  \( \mathcal{C}^{-k} \) non-trivially fibered over both \( \mathcal{T} \) and \( \mathcal{M}_c \)!
- What about \( \mathcal{Z}^{(k)}(\zeta, \tilde{\zeta}) \)?

Is the coupling well-defined?
The fivebrane partition function revisited

Key problem: worldvolume $W$ supports an (imaginary) self-dual 3-form $H = dB$

$$\star_W H = i H$$

[Callan, Curtis, Harvey, Strominger]

- The “flux” $H$ acts as an electric source for the 3-form $C$

- Non-chiral partition function is a sum over harmonic fluxes $H \in H^3(\mathcal{X}, \mathbb{Z})$

- Construct the chiral partition function via factorization [Witten]
Construct $Z$ by holomorphic factorization of the non-chiral partition function:

$$Z_{\text{non-chiral}}(C) = \sum_{H \in H^3(\mathcal{X},\mathbb{Z})} r(H) e^{-S(H,C)}$$

Gaussian action, weak-coupling approximation $g_s H << 1$

[References: Witten, Henningson, Nilsson, Salomonsson, Moore, Belov, Moore]
Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

$$
\mathcal{Z}_{\text{non-chiral}}(C) = \sum_{H \in H^3(\mathcal{X},\mathbb{Z})} r(H) e^{-S(H,C)}
$$

“fluctuation determinant” $r(H) = |\mathcal{F}|^2 \left[ \sigma \Theta(H) \right]^k$

metric-dependent normalization

[References:
Witten
Henningson, Nilsson, Salomonsson
Moore
Belov, Moore]
Construct $Z$ by holomorphic factorization of the non-chiral partition function:

$$Z_{\text{non-chiral}}(C) = \sum_{H \in H^3(X,Z)} r(H) e^{-S(H,C)}$$

“fluctuation determinant” $r(H) = |\mathcal{F}|^2 [\sigma_{\Theta}(H)]^k$

“quadratic refinement” of the intersection form on $H^3(X,Z)$

$$\sigma_{\Theta} : H^3(X,Z) \longrightarrow U(1)$$

Cocycle:

$$\sigma_{\Theta}(H + H') = (-1)^{\langle H, H' \rangle} \sigma_{\Theta}(H) \sigma_{\Theta}(H')$$
Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

$$
\mathcal{Z}^\text{non-chiral}(C) = \sum_{H \in H^3(\chi,\mathbb{Z})} r(H) e^{-S(H,C)}
$$

“fluctuation determinant” $r(H) = |\mathcal{F}|^2 \left[\sigma_\Theta(H)\right]^k$

General solution can be written as:

$$
\sigma_\Theta(H) = e^{-i\pi km_\Lambda n_\Lambda} + 2\pi i (m_\Lambda \theta^\Lambda - n_\Lambda \phi^\Lambda)
$$

$$
n^\Lambda = \int_{A^\Lambda} H \quad m_\Lambda = \int_{B^\Lambda} H
$$

“characteristics” may vary continuously over $\mathcal{M}_C$

integer-valued fluxes

[Witten]
[Henningson, Nilsson, Salomonsson]
[Moore][Belov,Moore]
Construct $Z$ by holomorphic factorization of the non-chiral partition function:

$$Z_{\text{non-chiral}}(C) = \sum_{H \in H^3(\mathcal{X},\mathbb{Z})} r(H) e^{-S(H,C)}$$

Choose a Lagrangian decomposition:

$$H^3(\mathcal{X},\mathbb{Z}) = \Gamma_e \oplus \Gamma_m$$

After Poisson resummation on $m_A \in \Gamma_m$

$$Z_{\text{non-chiral}}(C) \sim \sum_{\mu} \left| \mathcal{Z}_{\Theta,\mu}^{(k)}(\mathcal{N},\zeta,\tilde{\zeta}) \right|^2$$

Partition function of the chiral NS5-brane
Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

$$\mathcal{Z}^{\text{non-chiral}}(C) = \sum_{H \in H^3(\mathcal{X},\mathbb{Z})} r(H) e^{-S(H,C)}$$

Choose a Lagrangian decomposition: $H^3(\mathcal{X},\mathbb{Z}) = \Gamma_e \oplus \Gamma_m$

After Poisson resummation on $m_A \in \Gamma_m$

$$\mathcal{Z}^{\text{non-chiral}}(C) \sim \sum_{\mu} \left| \mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N},\zeta,\tilde{\zeta}) \right|^2$$

Period matrix in the “Weil complex structure” on $\mathcal{T}$
(determined by the Hodge star $\ast_{\mathcal{X}}$)
The chiral NS5-brane partition function \((E[x] = e^{2\pi i x})\)

\[
Z^{(k)}_{\Theta,\mu} = \mathcal{F} e^{\pi ik(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)} \sum_{n \in \Gamma_{m+\mu+\theta}} E \left[ \frac{k}{2} (\zeta^\Lambda - n^\Lambda)\tilde{N}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde{\zeta}_\Lambda - \phi_\Lambda)n^\Lambda \right]
\]

[Belov, Moore] (see also [Dijkgraaf, Verlinde, Vonk])

- **periodicity:** [Belov, Moore]

\[
Z^{(k)}_{\Theta,\mu}(\mathcal{N}, \zeta + n, \tilde{\zeta} + m) = (\sigma_\Theta(H))^k e^{\pi i(n^\Lambda \tilde{\zeta}_\Lambda - m^\Lambda \zeta^\Lambda)} Z^{(k)}_{\Theta,\mu}(\mathcal{N}, \zeta, \tilde{\zeta})
\]

is a holomorphic section of a line bundle \(\mathcal{L}^k_{\Theta} \rightarrow \mathcal{T}\) [Witten]

\[
\mathcal{D}\bar{\omega} Z^{(k)}_{\Theta,\mu} = 0 \quad \bar{\omega}_\Lambda = \tilde{\zeta}_\Lambda - \mathcal{N}_{\Lambda\Sigma}\zeta^\Sigma
\]
The chiral NS5-brane partition function \( (E[x] = e^{2\pi i x}) \)

\[
Z^{(k)}_{\Theta, \mu} = \mathcal{F} e^{\pi i k (\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \bar{\zeta}_\Lambda)} \sum_{n \in \Gamma_m + \mu + \theta} E \left[ \frac{k}{2} (\zeta^\Lambda - n^\Lambda) \bar{N}_{\Lambda \Sigma} (\zeta^\Sigma - n^\Sigma) + k (\bar{\zeta}_\Lambda - \phi_\Lambda) n^\Lambda \right]
\]

[Belov, Moore] (see also [Dijkgraaf, Verlinde, Vonk])

- **Periodicity:** [Belov, Moore]

\[
Z^{(k)}_{\Theta, \mu} (N, \zeta + n, \bar{\zeta} + m) = (\sigma_\Theta(H))^k e^{\pi i (n^\Lambda \bar{\zeta}_\Lambda - m^\Lambda \zeta^\Lambda)} Z^{(k)}_{\Theta, \mu} (N, \zeta, \bar{\zeta})
\]

→ \( \mu \) labels the \(|k| b_3(X)\) holomorphic sections of \( \mathcal{L}_\Theta^k \)

→ for a single NS5-brane \( k = 1 \) there is a unique holomorphic section of \( \mathcal{L}_\Theta \)
Back to the original problem of the coupling:

\[ ds^2_{\mathcal{M}} \big|_{\text{NS5}} \sim e^{-4\pi |k|} e^{\phi - i\pi k \sigma} Z^{(k)}_{\Theta, \mu} (\zeta, \tilde{\zeta}) \]

- \( e^{i\pi \sigma} \) is valued in \( \mathcal{C}(\phi) \) with \( c_1(\mathcal{C})\big|_\mathcal{T} = \omega_\mathcal{T} \)

- \( Z^{(1)}_{\Theta} \) is the unique holomorphic section of \( \mathcal{L}_{\Theta} \rightarrow \mathcal{T} \)

\[ c_1(\mathcal{L}_{\Theta}) = \omega_\mathcal{T} \]

→ NS5-brane instantons are well-defined under variations along \( \mathcal{T} \)
Back to the original problem of the coupling:

\[ ds^2_{\mathcal{M}} \bigg|_{\text{NS5}} \sim e^{-4\pi |k|} e^{\phi - i\pi k \sigma} Z^{(k)}_{\Theta, \mu} (\zeta, \tilde{\zeta}) \]

- What about \( \mathcal{L} \rightarrow \mathcal{M}_c \)?

- Part of the answer lies in the normalization factor \( \mathcal{F} \)
  
  \[ \text{[Belov, Moore]} \]

- \( \mathcal{F} \) is analogous to \( 1/\eta \) in the partition function of a 2d chiral boson
  
  \[ \text{[Alvarez-Gaume, Moore, Nelson, Vafa, Bost]} \]

Important to take proper account of supersymmetry

The correct coupling should be formulated as a deformation of the complex contact structure on twistor space
Twistor Space of $\mathcal{M}$ - a brief recap

$$\mathbb{CP}^1 \rightarrow \mathcal{Z}_M \rightarrow M$$

$\mathcal{Z}_M$ is a complex contact manifold with a Kähler-Einstein metric

contact one-form: $\chi^{[i]} = d\alpha^{[i]} + \xi_\Lambda^{[i]} d\tilde{\xi}_{\Lambda}^{[i]} - \tilde{\xi}^{[i]} d\xi_\Lambda^{[i]}$

[Salamon][LeBrun][Swann][de Wit, Rocek, Vandoren][Alexandrov, Saueressig, Pioline, Vandoren]
Twistor Space of $\mathcal{M}$ - a brief recap

\[ \mathbb{C}P^1 \longrightarrow \mathcal{Z}_\mathcal{M} \longrightarrow \mathcal{M} \]

$\mathcal{Z}_\mathcal{M}$ is a complex contact manifold with a Kähler-Einstein metric

contact one-form: $\mathcal{X}^{[i]} = d\alpha^{[i]} + \xi_\Lambda^{[i]} d\tilde{\xi}^{[i]}_\Lambda - \tilde{\xi}^{[i]}_\Lambda d\xi^{\Lambda}_{[i]}$

complex Darboux coordinates on $\mathcal{U}_i \subset \mathcal{Z}_\mathcal{M}$
Twistor Space of $\mathcal{M} - \text{a brief recap}$

$$\mathbb{CP}^1 \longrightarrow \mathcal{Z}_\mathcal{M} \longrightarrow \mathcal{M}$$

$\mathcal{Z}_\mathcal{M}$ is a complex contact manifold with a Kähler-Einstein metric

contact one-form: $\mathcal{X}^i = d\alpha^i + \xi^\Lambda d\tilde{\xi}^i - \tilde{\xi}^i d\xi^\Lambda$

Isometries of $\mathcal{M}$ lift to a holomorphic action on $\mathcal{Z}_\mathcal{M}:$

[Galicki][Salamon][Alexandrov, Saueressig, Pioline, Vandoren]

$\xi^\Lambda \longmapsto \xi^\Lambda + n^\Lambda$

$\tilde{\xi}_\Lambda \longmapsto \tilde{\xi}_\Lambda + m_\Lambda$

$\alpha \longmapsto \alpha + 2\kappa - m_\Lambda \xi^\Lambda + n^\Lambda \tilde{\xi}_\Lambda + c_\Theta(m, n)$
Twistor Space of $\mathcal{M}$ - a brief recap

$\mathbb{CP}^1 \rightarrow \mathcal{Z}_\mathcal{M} \rightarrow \mathcal{M}$

$\mathcal{Z}_\mathcal{M}$ is a complex contact manifold with a Kähler-Einstein metric

contact one-form: $\chi^i = d\alpha^i + \xi^\Lambda d\tilde{\xi}^i - \tilde{\xi}^i d\xi^\Lambda$

$(\xi^\Lambda, \tilde{\xi}_\Lambda)$ parametrize the complexified Jacobian torus:

$$\mathcal{T}^\mathbb{C} = \left[ \frac{H^3(\chi, \mathbb{R})}{H^3(\chi, \mathbb{Z})} \right]^\mathbb{C}$$

$e^{i\pi \alpha}$ transforms like a section of the “complexified” theta line bundle

$\mathbb{C}^\times \rightarrow \mathcal{L}^\mathbb{C}_\Theta \downarrow \mathcal{T}^\mathbb{C}$
Deformations of $\mathcal{M}$ can be uplifted to deformations of the complex contact structure on $\mathcal{Z}_\mathcal{M}$.

Infinitesimal deformations classified by $H^1(\mathcal{Z}_\mathcal{M}, \mathcal{O}(2))$

- Practically, we study transition functions $H^{ij}(\xi, \tilde{\xi}, \alpha)$ on overlapping patches.
- Specify the Darboux coordinates in terms of $(x, t) \in \mathcal{M} \times \mathbb{C}P^1$:

$$\xi^{\Lambda}_{[i]}(x, t), \quad \tilde{\xi}^{[i]}_{\Lambda}(x, t), \quad \alpha^{[i]}(x, t)$$

- Perturbative Darboux coordinates: [Alexandrov, Saueressig, Pioline, Vandoren]

$$\xi^{\Lambda} = \zeta^{\Lambda} + \frac{\tau_2}{2} \left( t^{-1} z^{\Lambda} - t \bar{z}^{\Lambda} \right)$$

$$\tilde{\xi}_{\Lambda} = \tilde{\zeta}_{\Lambda} + \frac{\tau_2}{2} \left( t^{-1} F_{\Lambda}(z) - t \bar{F}_{\Lambda}(\bar{z}) \right)$$

$$\alpha = \sigma + \frac{\tau_2}{2} \left( t^{-1} W(z) - t \bar{W}(\bar{z}) \right) + \frac{i \chi(\mathcal{X})}{24\pi} \log t$$

$\text{D}(-1)-\text{D}1$ \quad $\text{D}3-\text{D}5$ \quad $\text{NS}5$
Deformations of $\mathcal{M}$ can be uplifted to deformations of the complex contact structure on $\mathcal{Z}_\mathcal{M}$

→ Infinitesimal deformations classified by $H^1(\mathcal{Z}_\mathcal{M}, \mathcal{O}(2))$

- Practically, we study transition functions $H^{ij}(\xi, \tilde{\xi}, \alpha)$ on overlapping patches
- Specify the Darboux coordinates in terms of $(x, t) \in \mathcal{M} \times \mathbb{C}P^1$

$$\xi^\Lambda_{[i]}(x, t), \tilde{\xi}^{[i]}(x, t), \alpha^{[i]}(x, t)$$

- In the absence of NS5-branes, isometries of $\alpha$ are unbroken

→ $H^{ij}(\xi, \tilde{\xi}, \alpha) = H^{ij}(\xi, \tilde{\xi})$ reduce to symplectomorphisms

- When NS5-branes are present, $H^{ij}(\xi, \tilde{\xi}, \alpha)$ are genuine contact transformations
● When NS5-branes are present, $H^{[ij]}(\xi, \tilde{\xi}, \alpha)$ are genuine contact transformations.

Translations along $\mathcal{T}$ form a Heisenberg group with algebra:

$$[T_{\xi^\Lambda}, T_{\tilde{\xi}^\Sigma}] = k\delta^\Lambda_\Sigma$$

Cannot diagonalize $T_{\xi^\Lambda}$ and $T_{\tilde{\xi}^\Sigma}$ simultaneously

Must choose a polarization!

● $H^{[ij]}(\xi, \tilde{\xi}, \alpha)$ has a “non-abelian Fourier expansion”:

$$H^{[ij]}(\xi, \tilde{\xi}, \alpha) = \sum_{k \neq 0} \sum_{\mu \in (\Gamma_m/|k|)/\Gamma_m} \sum_{n \in \Gamma_m + \mu} H_{\mu,k}(\xi^\Lambda - n^\Lambda) E \left[ k n^\Lambda \tilde{\xi}^\Lambda - \frac{k}{2} \left( \tilde{\alpha} + \xi^\Lambda \tilde{\xi}^\Lambda \right) \right]$$

“Wave function”

[Ishikawa][Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline]
When NS5-branes are present, $H^{[ij]}(\xi, \tilde{\xi}, \alpha)$ are genuine contact transformations.

Translations along $\mathcal{T}$ form a Heisenberg group with algebra: 

$$[T_{\xi^\Lambda}, T_{\tilde{\xi}_\Sigma}] = k \delta_{\Sigma}^\Lambda$$

Cannot diagonalize $T_{\xi^\Lambda}$ and $T_{\tilde{\xi}_\Sigma}$ simultaneously.

Must choose a polarization!

$H^{[ij]}(\xi, \tilde{\xi}, \alpha)$ has a “non-abelian Fourier expansion”:

$$H^{[ij]}(\xi, \tilde{\xi}, \alpha) = \sum_{k \neq 0} \sum_{\mu \in (\Gamma_m / |k|) / \Gamma_m} \sum_{n \in \Gamma_m + \mu} H_{\mu,k}(\xi^\Lambda - n^\Lambda) \mathbf{E} \left[ k n^\Lambda \tilde{\xi}_\Lambda - \frac{k}{2} (\tilde{\alpha} + \xi^\Lambda \tilde{\xi}_\Lambda) \right]$$

“Wave function”

Suggestive relation with the topological string...
NS5-instantons in twistor space

Aim: Find the twistor space analogue of the fivebrane partition function

General arguments require:

\[ H_k(\xi, \tilde{\xi}, \alpha) \sim e^{-i\pi k \alpha} Z^{(k)}(\xi, \tilde{\xi}) \]
NS5-instantons in twistor space

Aim: Find the twistor space analogue of the fivebrane partition function

“Exploratory strategy”: Start from the D-instanton series in IIB and employ S-duality

\[(D5, \text{NS5}) \text{ form a doublet under } SL(2, \mathbb{Z})\]

• Similar philosophy as was previously done for \((F1,D1) + D(-1)\)

[Robles-Llana, Rocek, Saueressig, Theis, Vandoren]
NS5-instantons in twistor space

Aim: Find the twistor space analogue of the fivebrane partition function

“Exploratory strategy”: Start from the D-instanton series in IIB and employ S-duality

\[ \text{(D5, NS5) form a doublet under } \text{SL}(2, \mathbb{Z}) \]

- Deformations arise due to D(-1)-D1-D3-D5 wrapping

\[ \gamma = p^0 + p^a \omega_a - q_a \omega^a + q_0 \omega^{\hat{X}} \in H_{\text{even}}(\hat{X}, \mathbb{Z}) \]

- “BPS ray” \( \ell_\gamma = \{ t : Z_\gamma(z^a)/t \in i\mathbb{R}^- \} \)

- Across each ray, \( (\xi^\Lambda, \tilde{\xi}_\Lambda) \) are related by a symplectomorphism

\[ H_\gamma = \frac{i}{2(2\pi)^2} \Omega(\gamma) \text{Li}_2 \left[ \sigma_\Theta(\gamma) e^{-2\pi i (q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)} \right] \]

generalized DT-invariants

same QR as in the fivebrane partition function!
NS5-instantons in twistor space

Aim: Find the twistor space analogue of the fivebrane partition function

“Exploratory strategy”: Start from the D-instanton series in IIB and employ S-duality

\((D5, NS5)\) form a doublet under \(SL(2, \mathbb{Z})\)

- Deformations arise due to D(-1)-D1-D3-D5 wrapping

\[\gamma = p^0 + p^a \omega_a - q_a \omega^a + q_0 \omega \hat{\chi} \in H_{even}(\hat{\chi}, \mathbb{Z})\]

- “BPS ray” \(\ell_\gamma = \{ t : Z_\gamma(z^a) / t \in i\mathbb{R}^- \}\)

- Across each ray, \((\xi_\Lambda, \tilde{\xi}_\Lambda)\) are related by a symplectomorphism

\[H_\gamma = \frac{i}{2(2\pi)^2} \tilde{\Omega}(\gamma) \sigma \Theta(\gamma) e^{-2\pi i (q_\Lambda \xi_\Lambda - p^\Lambda \tilde{\xi}_\Lambda)}\]

\[\tilde{\Omega}(\gamma) = \sum_{d | \gamma} \frac{1}{d^2} \Omega(\gamma / d)\]

[Kontsevich, Soibelman][Gaiotto, Moore, Neitzke] [Alexandrov, Saueressig, Pioline, Vandoren]
S-duality

- **Data:** \((H_\gamma, \ell_\gamma)\) \quad \Rightarrow \quad \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\hat{\gamma}}, \ell_{k,p,\hat{\gamma}})

\[
\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
\]

- \(\gcd(c, d) = 1\)

- \((c, d) = (-k, p)/p^0\) \quad \text{NS5-charge} \quad k \quad p^0 = \gcd(k, p)

- \(p\) \quad \text{D5-charge}

- **Reduced charge vector:** \(\hat{\gamma} = (p^a, \hat{q}_a, \hat{q}_0)\)
S-duality

- **Data:** \((H_\gamma, \ell_\gamma) \rightarrow \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\gamma}, \ell_{k,p,\gamma})\)

- **Key property:** 
  \[
  \begin{pmatrix}
  \tilde{\xi}_0 \\
  \alpha
  \end{pmatrix}
  \mapsto
  \begin{pmatrix}
  d & -c \\
  -b & a
  \end{pmatrix}
  \begin{pmatrix}
  \tilde{\xi}_0 \\
  \alpha
  \end{pmatrix}
  + \cdots
  \]

  \[
  \tilde{\xi}_0 = \tilde{\xi}_0 + \frac{\tau_2}{2} \left( t^{-1} F_\Lambda(z) - t \bar{F}_\Lambda(\bar{z}) \right)
  \]

  \[
  \alpha = \sigma + \frac{\tau_2}{2} \left( t^{-1} W(z) - t \bar{W}(\bar{z}) \right) + \frac{i\chi(\mathcal{X})}{24\pi} \log t
  \]

- **Main assumption:** S-duality leaves the complex contact structure invariant
  - May be questionable, but works for D(-1)-(F1,D1)

[Alexandrov, Saueressig, Pioline, Vandoren][Alexandrov, Saueressig]
S-duality

- **Data:** \((H_\gamma, \ell_\gamma)\) \rightarrow \{S-\}\: (H_\gamma, \ell_\gamma) = (H_{k,p,\tilde{\gamma}}, \ell_{k,p,\tilde{\gamma}})

- **After some trickery:** \((E[x] = e^{2\pi i x})\)

\[
H_{k,p,\hat{\gamma}} = \frac{i \sigma \Theta(\gamma)}{8 \pi^2} \tilde{\Omega}(\gamma) E \left[ -\frac{k}{2} S_\alpha + \frac{p^0(k \hat{q}_a (\xi^a - n^a) + p^0 \hat{q}_0)}{k^2(\xi^0 - n^0)} \right]
\]

\[
S_\alpha \equiv \alpha + (\xi^\Lambda - 2n^\Lambda)\tilde{\xi}_\Lambda + 2\frac{N(\xi^a - n^a)}{\xi^0 - n^0}
\]

\[
(n^0, n^a) = (p, p^a)/k
\]

\[
N(\xi^a) \equiv \frac{1}{6} \kappa_{abc} \xi^a \xi^b \xi^c
\]
S-duality

- Data: \((H_\gamma, \ell_\gamma) \rightarrow \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\hat{\gamma}}, \ell_{k,p,\hat{\gamma}})\)

- After some trickery: \((E[x] = e^{2\pi i x})\)

\[
H_{k,p,\hat{\gamma}} = \frac{i\sigma \Theta(\gamma)}{8\pi^2} \tilde{\Omega}(\gamma) \mathbb{E} \left[ -\frac{k}{2} S_\alpha + \frac{p^0(k\hat{q}_a(\xi^a - n^a) + p^0\hat{q}_0)}{k^2(\xi^0 - n^0)} \right]
\]

\[
S_\alpha \equiv \alpha + (\xi^\Lambda - 2n^\Lambda)\tilde{\xi}_\Lambda + 2\frac{N(\xi^a - n^a)}{\xi^0 - n^0}
\]

- Up to subtle phases, the set of data \((\ell_{k,p,\hat{\gamma}}, H_{k,p,\hat{\gamma}})\) is invariant under Heisenberg translations of \((\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)\)

- Assuming that \(\tilde{\Omega}(\gamma)\) is invariant under “generalized spectral flow”
Find the analogue of the fivebrane partition function

Consider the formal sum:

\[
H_{\text{NS5}}^{(k)}(\xi, \bar{\xi}^\Lambda, \alpha) = \sum_{p, p^a, q_\Lambda} H_{k, p, \hat{\gamma}}(\xi, \bar{\xi}^\Lambda, \alpha)
\]
Use Heisenberg invariance to rewrite:

\[ H_{\text{NS5}}^{(k)}(\xi, \tilde{\xi}, \alpha) = \sum_{\mu \in (\Gamma_m/|k|)/\Gamma_m} \sum_{n \in \Gamma_m + \mu} H_{\text{NS5}}^{(k, \mu)}(\xi^\Lambda - n^\Lambda) E \left[ k n^\Lambda (\tilde{\xi}^\Lambda - \phi^\Lambda) - \frac{k}{2} (\tilde{\alpha} + \xi^\Lambda \tilde{\xi}^\Lambda) \right] \]

“NS5 wave function”

- This is a non-Gaussian theta series

Twistor space analogue of the chiral fivebrane partition function!

Does this make sense?
Restrict to \( k = 1 \) → Considerable simplification:

\[
H^{(1,0)}_{\text{NS5}}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi^0}} + \pi A_\Lambda \xi^\Lambda \xi^\Sigma \sum_{\hat{q}_a, \hat{q}_0} \tilde{\Omega}(\gamma) (-1)^{\hat{q}_0} e^{2\pi i \hat{q}_a \xi^a (\xi^0)^{-1} + 2\pi i \hat{q}_0 (\xi^0)^{-1}}
\]

- Now compare with (rank 1) D6-D2-D0 DT-partition function:

\[
\mathcal{Z}_{\text{DT}} = \sum_{Q_a, J} (-1)^{2J} N_{\text{DT}}(Q_a, 2J) e^{-2\lambda J + 2\pi i Q_a z^a}
\]

D2-charge

D0-charge
Considerable simplification:

\[ H^{(1,0)}_{\text{NS5}}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi_0} + \pi A_{\Lambda \Sigma} \xi^\Lambda \xi^\Sigma} \sum_{\hat{q}_a, \hat{q}_0} \tilde{\Omega}(\gamma)(-1)^{\hat{q}_0} e^{2\pi i \hat{q}_a \xi^a (\xi^0)^{-1}} + 2\pi i \hat{q}_0 (\xi^0)^{-1} \]

- Now compare with (rank 1) D6-D2-D0 DT-partition function:

\[ Z_{\text{DT}} = \sum_{Q_a, J} (-1)^{2J} N_{\text{DT}}(Q_a, 2J) e^{-2\lambda J + 2\pi i Q_a z^a} \]

- Under formal change of variables: \((\lambda, z^a) = (2\pi / (i\xi^0), \xi^a / \xi^0)\)

\((Q_a, 2J) = (\hat{q}_a + c_{2,a}/24, \hat{q}_0)\)
Restrict to \( k = 1 \)  

Considerable simplification:

\[
H^{(1,0)}_{\text{NS5}}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi_0}} + \pi A_\Lambda \Sigma \xi^\Lambda \xi^\Sigma \sum_{\hat{q}_a, \hat{q}_0} \tilde{\Omega}(\gamma) (-1) \hat{q}_0 e^{2\pi i \hat{q}_a \xi^a (\xi^0)^{-1} + 2\pi i \hat{q}_0 (\xi^0)^{-1}}
\]

Correct sign factor reproduced from the quadratic refinement!

\[
Z_{\text{DT}} = \sum_{Q_a, J} (-1)^{2J} N_{\text{DT}}(Q_a, 2J) e^{-2\lambda J + 2\pi i Q_a z^a}
\]

- Under formal change of variables:  
  \[
  (\lambda, z^a) = \left( \frac{2\pi}{i \xi^0}, \frac{\xi^a}{\xi^0} \right)
  \]
  
  \[
  (Q_a, 2J) = (\hat{q}_a + c_{2,a}/24, \hat{q}_0)
  \]

\[
H^{(1,0)}_{\text{NS5}}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi_0}} + \pi A_\Lambda \Sigma \xi^\Lambda \xi^\Sigma Z_{\text{DT}}(\xi^\Lambda)
\]
Actually, a more accurate statement is:

$$H_{\text{NS5}}^{(1,0)}(\xi^\Lambda) \sim \Psi_{\mathbb{R}}^{\text{top}}(\xi)$$

Topological string amplitude in the “real” (background-independent) polarization

[Witten][Maulik, Nekrasov, Okounkov, Pandharipande][Denef, Moore][Schwarz, Tang]
Actually, a more accurate statement is:

\[ H^{(1,0)}_{\text{NS5}}(\xi^\Lambda) \sim \Psi_{\text{top}}^{\text{R}}(\xi) \]

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

\[ H^{(1)}_{\text{NS5}}(\xi, \tilde{\xi}, \alpha) = e^{-i\pi\alpha} \sum_{n \in \Gamma_m} \Psi_{\text{top}}^{\text{R}}(\xi^\Lambda - n^\Lambda) \mathbf{E} \left[ n^\Lambda \tilde{\xi}_\Lambda - \frac{1}{2} \xi^\Lambda \tilde{\xi}_\Lambda \right] \]

Fits nicely with earlier speculations: [Dijkgraaf, Verlinde, Vonk][Nekrasov, Ooguri, Vafa][Kapustin]
Actually, a more accurate statement is:

\[ H^{(1,0)}_{\text{NS5}}(\xi^\Lambda) \sim \Psi^\text{top}_\mathbb{R}(\xi) \]

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

\[ H^{(1)}_{\text{NS5}}(\xi, \tilde{\xi}, \alpha) \equiv e^{-i\pi\alpha} \mathcal{Z}^{(1)}(\xi, \tilde{\xi}) \]

- Now reexamine the NS5-coupling:

  \[ \mathcal{Z}^{(1)} \quad \text{and} \quad e^{i\pi\alpha} \quad \text{transform as a sections of} \quad \mathcal{L}^\mathbb{C}_\Theta \rightarrow \mathcal{T}^\mathbb{C} \quad \text{Ok!} \]

  \[ \rightarrow \quad \text{Both factors are separately trivial under} \quad \mathcal{L} \rightarrow \mathcal{M}_c \quad \text{Ok!} \]

  Follows from: \[ \alpha = \sigma + \frac{\tau_2}{2} \left( t^{-1}W(z) - t \bar{W}(\bar{z}) \right) + \frac{i\chi(\mathcal{X})}{24\pi} \log t \]
Actually, a more accurate statement is:

\[ H_{\text{NS5}}^{(1,0)}(\xi^\Lambda) \sim \Psi^{\text{top}}_{\mathbb{R}}(\xi) \]

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

\[ H_{\text{NS5}}^{(1)}(\xi, \tilde{\xi}, \alpha) \equiv e^{-i\pi\alpha} \mathcal{Z}^{(1)}(\xi, \tilde{\xi}) \]

Coupling is well-defined!
Actually, a more accurate statement is:

\[ H_{NS5}^{(1,0)}(\xi^A) \sim \psi_{\text{top}}(\xi) \]

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

\[ H_{NS5}^{(1)}(\xi, \tilde{\xi}, \alpha) \equiv e^{-i\pi\alpha} Z^{(1)}(\xi, \tilde{\xi}) \]

The contribution from \( k > 1 \) NS5-branes should be given by the generating function of rank \( r = \gcd(k, p) \) DT-invariants
But what about the Gaussian type IIA partition function?

By mirror symmetry this is given by a theta series based on the B-model topological string amplitude.
But what about the Gaussian type IIA partition function?

- **Consistency check**: project onto the base $\mathcal{M}$ via the Penrose transform

\[ e^{-i\pi\sigma} Z^{(1)}(\mathcal{N}, \zeta^\Lambda, \tilde{\zeta}^\Lambda) = \sum_{n \in \Gamma_m} \int_{\mathcal{C}} \frac{dt}{2\pi it} e^{-i\pi\alpha} Z^{(1)}(\xi, \tilde{\xi}) \]

\[ \sim e^{-i\pi\sigma} e^{f_1(z)} \sum_{n \in \Gamma_m} [(\zeta^\Lambda - n^\Lambda)(\bar{\zeta}^\Lambda \mathcal{N}_{\Lambda\Sigma}) z^\Sigma]^{-\frac{1}{24} - 1} e^{-S_{\text{Gauss}}^{\text{IIA}}(\mathcal{N}, \zeta, \tilde{\zeta})} \]

→ **Saddle point approx. reproduces Gaussian partition function with insertion**

→ **Phase of the normalization factor determined:**

\[ \mathcal{F} \sim e^{f_1(z)} \]

\[ e^{f_1}: \text{holomorphic part of the B-model one-loop amplitude} \quad F_1 = \log \left[ e^{f_1(z)} + \bar{f}_1(\bar{z}) / \sqrt{M(z, \bar{z})} \right] \]
Summary and Discussion

- Initial steps towards understanding NS5-brane instantons in type II/CY3

  → Proposal for the normalization factor of (non-)chiral partition function

  → Non-linear partition function in twistor space

  → Proposal for contact transformation encoding NS5-deformations

  → Explicit relation with topological string wave functions and DT-invariants
Summary and Discussion

- Crucial open problem: understand wall crossing

- KS-GMN symplectomorphisms should be upgraded to contact transformations

- What about the motivic KSWCF?

Here is a suggestive hint: consider the Heisenberg group elements

\[ \gamma = (n_{\Lambda}, m_{\Lambda}) \quad T_{\gamma} : \quad \xi_{\Lambda} \rightarrow \xi_{\Lambda} + n_{\Lambda} \]

\[ \tilde{\xi}_{\Lambda} \rightarrow \tilde{\xi}_{\Lambda} + m_{\Lambda} \]

\[ \alpha \rightarrow \alpha - m_{\Lambda} \xi_{\Lambda} + n_{\Lambda} \tilde{\xi}_{\Lambda} \]
Summary and Discussion

- Crucial open problem: understand wall crossing

KS-GMN symplectomorphisms should be upgraded to contact transformations

What about the motivic KSWCF?

Here is a suggestive hint: consider the Heisenberg group elements

When acting on sections of the form

$$F_k(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) \equiv F_k(\xi^\Lambda, \tilde{\xi}_\Lambda) e^{-i\pi k \alpha}$$

$$T_{\gamma} T_{\gamma'} = q^{1/2} \langle \gamma, \gamma' \rangle T_{\gamma + \gamma'}$$

quantum torus with deformation parameter:

$$q^{1/2} = -e^{i\pi k}$$

Quantum deformation ↔ NS5-brane charge
Summary and Discussion

- Crucial open problem: understand **wall crossing**

  - **KS-GMN symplectomorphisms should be upgraded to contact transformations**

  - **What about the motivic KSWCF?**

Here is a suggestive hint: consider the Heisenberg group elements

When acting on sections of the form 

$$F_k(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) \equiv F_k(\xi^\Lambda, \tilde{\xi}_\Lambda) e^{-i\pi k\alpha}$$

Is there a “generalized” classical limit $q^{1/2} \rightarrow -e^{i\pi k}$ where the quantum dilogarithm reduces to a contact transformation?