Fivebrane Instantons and Hypermultiplets

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“Advances in string theory, wall crossing and quaternion-Kähler geometry”

IHP, Paris, Aug-Sept 2010

Based on [Alexandrov, D.P. , Pioline] to appear

Also related: [Pioline, D.P., 0902.3274]
[Bao, Kleinschmidt, Nilsson, D.P., Pioline, 0909.4299 & 1005.4848]

So far: perturbative and D-instanton corrections to the hypermultiplet metric

This talk: take initial steps towards completing the picture

$\longrightarrow$ include NS5-branes

[Becker, Becker, Strominger]

$\longrightarrow$ include NS5-branes

IIA

– worldvolume theory is chiral

– partition function is a section of a line bundle over the space of 3-forms

– requires a choice of “quadratic refinement”

IIB

– worldvolume theory is non-chiral

– related to D5 by S-duality

– DT-invariants and topological strings

– non-trivial contact structure on twistor space $\mathcal{Z}_{\mathcal{M}} \;\longrightarrow\; \mathcal{M}$

$\boxed{\text{How to implement NS5-corrections to } \mathcal{M}\,?}$

Outline

- The HM moduli space in IIA/CY3
- The type IIA fivebrane partition function revisited
- Uplifting to twistor space
- NS5-brane instantons from S-duality
- Conclusions and Discussion

Type IIA point of view on $\mathcal{M}$ (recap from Boris's talk)

In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian

$$\mathcal{M} \sim \mathbb{R}^+_\phi \times \left(S^1_\sigma \;\longrightarrow\; \begin{array}{c} \mathcal{C}(\phi) \\ \downarrow \\ \mathcal{J}_c(\mathcal{X}) \end{array} \right) \qquad \dim \mathcal{M} = 4(h_{2,1}+1)$$

$g_s = e^{-\phi/2}$

σ

dual of $B_{(2)}$ **in** $D=4$

"NS-axion"

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$$\begin{array}{ccc} H^3(\mathcal{X},\mathbb{R})/H^3(\mathcal{X},\mathbb{Z}) & \to & \mathcal{J}_c(\mathcal{X}) \\ & & \downarrow \\ & & \mathcal{M}_c(\mathcal{X}), \end{array}$$

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C_{(3)} \qquad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C_{(3)}$$

$$(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in \frac{H^3(\mathcal{X},\mathbb{R})}{H^3(\mathcal{X},\mathbb{Z})} = \mathcal{T}$$

$$\dim \mathcal{T} = 2h_{2,1} + 2$$

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$$H^3(\mathcal{X},\mathbb{R})/H^3(\mathcal{X},\mathbb{Z}) \longrightarrow \mathcal{J}_c(\mathcal{X})$$

$$\downarrow$$

$$\mathcal{M}_c(\mathcal{X}), \qquad \dim \mathcal{M}_c = 2h_{2,1}$$

$$\zeta^\Lambda = \int_{\mathcal{A}^\Lambda} C_{(3)} \qquad \tilde{\zeta}_\Lambda = \int_{\mathcal{B}_\Lambda} C_{(3)}$$

$$(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in \frac{H^3(\mathcal{X},\mathbb{R})}{H^3(\mathcal{X},\mathbb{Z})} = \mathcal{T}$$

$$X^I = \int_{\mathcal{A}^I} \Omega$$

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First Chern class: $c_1(\mathcal{C}) = d\left(\frac{D\sigma}{2}\right) = \omega_\mathcal{T} + \frac{\chi}{24}\omega_{\mathcal{M}_c}$

classical contribution

one-loop correction
χ Euler number of $\mathcal{X}$

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Translations along $\mathcal{T}$ form a Heisenberg group

$$\begin{aligned} \zeta^\Lambda &\longmapsto \zeta^\Lambda + n^\Lambda \\ \tilde{\zeta}_\Lambda &\longmapsto \tilde{\zeta}_\Lambda + m_\Lambda \\ \sigma &\longmapsto \sigma + 2\kappa - m_\Lambda \zeta^\Lambda + n^\Lambda \tilde{\zeta}_\Lambda \end{aligned}$$

broken by D2-instantons

$n^\Lambda, m_\Lambda \in \mathbb{Z}$

broken by NS5-instantons

$\kappa \in \mathbb{Z}$

Type IIA point of view on $\mathcal{M}$ (recap from Boris's talk)

→ **In the weak-coupling limit, the moduli space is foliated by circle fibrations over the intermediate Jacobian**

$$\mathcal{M} \sim \mathbb{R}^+_\phi \times \begin{pmatrix} S^1_\sigma & \to & \mathcal{C}(\phi) \\ & & \downarrow \\ & & \mathcal{J}_c(\mathcal{X}) \end{pmatrix}$$

Translations along $\mathcal{T}$ form a Heisenberg group

$$\zeta^\Lambda \longmapsto \zeta^\Lambda + n^\Lambda$$

$$\tilde{\zeta}_\Lambda \longmapsto \tilde{\zeta}_\Lambda + m_\Lambda$$

$$\sigma \longmapsto \sigma + 2\kappa - m_\Lambda \zeta^\Lambda + n^\Lambda \tilde{\zeta}_\Lambda + c_\Theta(m,n)$$

cocycle

required for consistency of NS5-instantons

Qualitative form of NS5-instanton corrections

$$ds^2_{\mathcal{M}}\big|_{\rm NS5} \sim e^{-4\pi|k|\, e^{\phi} - i\pi k\sigma}\; \mathcal{Z}^{(k)}(\zeta, \tilde\zeta)$$

chiral NS5-partition function

Qualitative form of NS5-instanton corrections

$$ds^2_{\mathcal{M}}\big|_{\mathrm{NS5}} \sim e^{-4\pi|k|\,e^{\phi}-i\pi k\sigma}\,\mathcal{Z}^{(k)}(\zeta,\tilde{\zeta})$$

- NS5-instanton action: $S_{\mathrm{NS5}} = 4\pi|k|\left(g_s^{-2}+\cdots\right) + i\pi\left(\sigma+\cdots\right)$
- $e^{-i\pi k\sigma}$ is valued in the circle bundle $\mathcal{C}^{-k}$

$\longrightarrow$ $\mathcal{C}^{-k}$ non-trivially fibered over both $\mathcal{T}$ and $\mathcal{M}_c$!

- What about $\mathcal{Z}^{(k)}(\zeta,\tilde{\zeta})$?

$\boxed{\text{Is the coupling well-defined?}}$

The fivebrane partition function revisited

Key problem: worldvolume W supports an (imaginary) self-dual 3-form $H = d\mathcal{B}$

$$\star_W H = iH$$

[Callan, Curtis, Harvey, Strominger]

- The "flux" H acts as an electric source for the 3-form C

- Non-chiral partition function is a sum over harmonic fluxes $H \in H^3(\mathcal{X}, \mathbb{Z})$

- Construct the chiral partition function via factorization [Witten]

Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

$$\mathcal{Z}^{\text{non-chiral}}(C) = \sum_{H \in H^3(\mathcal{X},\mathbb{Z})} r(H)\, e^{-S(H,C)}$$

[Witten]
[Henningson, Nilsson, Salomonsson]
[Moore][Belov,Moore]

Gaussian action, weak-coupling approximation $g_s H << 1$

Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

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"fluctuation determinant" $r(H) = |\mathcal{F}|^2 \, \left[\sigma_\Theta(H)\right]^k$

metric-dependent normalization

Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

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"fluctuation determinant" $r(H) = |\mathcal{F}|^2 \; [\sigma_\Theta(H)]^k$

"quadratic refinement" of the intersection form on $H^3(\mathcal{X}, \mathbb{Z})$

$$\sigma_\Theta \;:\; H^3(\mathcal{X}, \mathbb{Z}) \;\longrightarrow\; U(1)$$

cocycle: $$\sigma_\Theta(H + H') = (-1)^{\langle H, H' \rangle} \sigma_\Theta(H) \sigma_\Theta(H')$$

Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

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[Moore][Belov,Moore]

"fluctuation determinant" $r(H) = |\mathcal{F}|^2\ [\sigma_\Theta(H)]^k$

general solution can be written as:

$$\sigma_\Theta(H) = e^{-i\pi k m_\Lambda n^\Lambda + 2\pi i (m_\Lambda \theta^\Lambda - n^\Lambda \phi_\Lambda)}$$

$$n^\Lambda = \int_{\mathcal{A}^\Lambda} H \qquad m_\Lambda = \int_{\mathcal{B}_\Lambda} H \qquad \Theta = (\theta^\Lambda, \phi_\Lambda)$$

integer-valued fluxes

"characteristics"
may vary continuously over $\mathcal{M}_C$

Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

$$\mathcal{Z}^{\text{non-chiral}}(C) = \sum_{H \in H^3(\mathcal{X},\mathbb{Z})} r(H)\, e^{-S(H,C)}$$

Choose a Lagrangian decomposition: $H^3(\mathcal{X},\mathbb{Z}) = \Gamma_e \oplus \Gamma_m$

After Poisson resummation on $m_\Lambda \in \Gamma_m$

$$\mathcal{Z}^{\text{non-chiral}}(C) \sim \sum_{\mu} \left| \mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N},\zeta,\tilde{\zeta}) \right|^2$$

Partition function of the chiral NS5-brane

Construct $\mathcal{Z}$ by holomorphic factorization of the non-chiral partition function:

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Period matrix in the "Weil complex structure" on $\mathcal{T}$

(determined by the Hodge star $\star_{\mathcal{X}}$)

The chiral NS5-brane partition function $(\mathbf{E}[x] = e^{2\pi i x})$

$$\mathcal{Z}^{(k)}_{\Theta,\mu} = \mathcal{F} e^{\pi i k(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)} \sum_{n \in \Gamma_m + \mu + \theta} \mathbf{E}\Big[\frac{k}{2}(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde{\zeta}_\Lambda - \phi_\Lambda) n^\Lambda\Big]$$

[Belov, Moore] (see also [Dijkgraaf, Verlinde, Vonk])

- periodicity: [Belov, Moore]

$$\mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, \zeta + n, \tilde{\zeta} + m) = (\sigma_\Theta(H))^k \, e^{\pi i (n^\Lambda \tilde{\zeta}_\Lambda - m_\Lambda \zeta^\Lambda)} \mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, \zeta, \tilde{\zeta})$$

$\longrightarrow$ is a holomorphic section of a line bundle $\mathcal{L}^k_\Theta \longrightarrow \mathcal{T}$ [Witten]

$$\mathcal{D}_{\bar{\omega}} \mathcal{Z}^{(k)}_{\Theta,\mu} = 0 \qquad \bar{\omega}_\Lambda = \tilde{\zeta}_\Lambda - \mathcal{N}_{\Lambda\Sigma}\zeta^\Sigma$$

The chiral NS5-brane partition function $(\mathbf{E}[x] = e^{2\pi i x})$

$$\mathcal{Z}^{(k)}_{\Theta,\mu} = \mathcal{F} e^{\pi i k(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde\zeta_\Lambda)} \sum_{n\in\Gamma_m+\mu+\theta} \mathbf{E}\left[\frac{k}{2}(\zeta^\Lambda - n^\Lambda)\bar{\mathcal{N}}_{\Lambda\Sigma}(\zeta^\Sigma - n^\Sigma) + k(\tilde\zeta_\Lambda - \phi_\Lambda)n^\Lambda\right]$$

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$$\mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, \zeta + n, \tilde\zeta + m) = (\sigma_\Theta(H))^k \, e^{\pi i (n^\Lambda \tilde\zeta_\Lambda - m_\Lambda \zeta^\Lambda)} \, \mathcal{Z}^{(k)}_{\Theta,\mu}(\mathcal{N}, \zeta, \tilde\zeta)$$

$\longrightarrow$ μ labels the $|k|^{b_3(\mathcal{X})}$ holomorphic sections of $\mathcal{L}^k_\Theta$

$\longrightarrow$ for a single NS5-brane $k=1$ there is a unique holomorphic section of $\mathcal{L}_\Theta$

Back to the original problem of the coupling:

$$ds^2_{\mathcal{M}}\Big|_{\mathrm{NS5}} \sim e^{-4\pi|k|\,e^{\phi}-i\pi k\sigma}\;\mathcal{Z}^{(k)}_{\Theta,\mu}(\zeta,\tilde{\zeta})$$

- $e^{i\pi\sigma}$ is valued in $\mathcal{C}(\phi)$ with $c_1(\mathcal{C})\Big|_{\mathcal{T}} = \omega_{\mathcal{T}}$

- $\mathcal{Z}^{(1)}_{\Theta}$ is the unique holomorphic section of $\mathcal{L}_{\Theta} \longrightarrow \mathcal{T}$

$$c_1(\mathcal{L}_{\Theta}) = \omega_{\mathcal{T}}$$

$\longrightarrow$ NS5-brane instantons are well-defined under variations along $\mathcal{T}$

Back to the original problem of the coupling:

$$ds^2_{\mathcal{M}}\Big|_{\rm NS5} \sim e^{-4\pi|k|\,e^{\phi}-i\pi k\sigma}\, \mathcal{Z}^{(k)}_{\Theta,\mu}(\zeta,\tilde{\zeta})$$

- What about $\mathcal{L} \longrightarrow \mathcal{M}_c$?

$\longrightarrow$ Part of the answer lies in the normalization factor $\mathcal{F}$

[Belov, Moore]

- $\mathcal{F}$ is analogous to $1/\eta$ in the partition function of a 2d chiral boson

[Alvarez-Gaume, Moore, Nelson, Vafa, Bost]

Important to take proper account of supersymmetry

The correct coupling should be formulated as a deformation of the complex contact structure on twistor space

Twistor Space of $\mathcal{M}$ - a brief recap

$$\mathbb{C}P^1 \longrightarrow \mathcal{Z}_{\mathcal{M}} \longrightarrow \mathcal{M}$$

$\mathcal{Z}_{\mathcal{M}}$ is a complex contact manifold with a Kähler-Einstein metric

contact one-form: $$\mathcal{X}^{[i]} = d\alpha^{[i]} + \xi^{\Lambda}_{[i]} d\tilde{\xi}^{[i]}_{\Lambda} - \tilde{\xi}^{[i]}_{\Lambda} d\xi^{\Lambda}_{[i]}$$

[Salamon][LeBrun][Swann][de Wit, Rocek, Vandoren][Alexandrov, Saueressig, Pioline, Vandoren]

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complex Darboux coordinates on $\mathcal{U}_i \subset \mathcal{Z}_{\mathcal{M}}$

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Isometries of $\mathcal{M}$ lift to a holomorphic action on $\mathcal{Z}_{\mathcal{M}}$:

[Galicki][Salamon][Alexandrov, Saueressig, Pioline, Vandoren]

$$\xi^{\Lambda} \longmapsto \xi^{\Lambda} + n^{\Lambda}$$

$$\tilde{\xi}_{\Lambda} \longmapsto \tilde{\xi}_{\Lambda} + m_{\Lambda}$$

$$\alpha \longmapsto \alpha + 2\kappa - m_{\Lambda}\xi^{\Lambda} + n^{\Lambda}\tilde{\xi}_{\Lambda} + c_{\Theta}(m,n)$$

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$\longrightarrow$ $(\xi^{\Lambda}, \tilde{\xi}_{\Lambda})$ parametrize the complexified Jacobian torus:

$$\mathcal{T}^{\mathbb{C}} = \left[\frac{H^3(\mathcal{X}, \mathbb{R})}{H^3(\mathcal{X}, \mathbb{Z})}\right]^{\mathbb{C}}$$

$\longrightarrow$ $e^{i\pi\alpha}$ transforms like a section of the "complexified" theta line bundle

$$\begin{array}{ccc} \mathbb{C}^{\times} \longrightarrow & & \mathcal{L}^{\mathbb{C}}_{\Theta} \\ & & \downarrow \\ & & \mathcal{T}^{\mathbb{C}} \end{array}$$

Deformations of $\mathcal{M}$ can be uplifted to deformations of the complex contact structure on $\mathcal{Z}_{\mathcal{M}}$ [Salamon][LeBrun]

→ Infinitesimal deformations classified by $H^1(\mathcal{Z}_{\mathcal{M}}, \mathcal{O}(2))$

- Practically, we study transition functions $H^{[ij]}(\xi, \tilde{\xi}, \alpha)$ on overlapping patches
- Specify the Darboux coordinates in terms of $(x,t) \in \mathcal{M} \times \mathbb{C}P^1$

$$\xi^{\Lambda}_{[i]}(x,t), \;\; \tilde{\xi}^{[i]}_{\Lambda}(x,t), \;\; \alpha^{[i]}(x,t)$$

- Perturbative Darboux coordinates: [Alexandrov, Saueressig, Pioline, Vandoren]

$$\xi^{\Lambda} = \zeta^{\Lambda} + \frac{\tau_2}{2}\left(t^{-1} z^{\Lambda} - t\, \bar{z}^{\Lambda}\right)$$ D(-1)-D1

$$\tilde{\xi}_{\Lambda} = \tilde{\zeta}_{\Lambda} + \frac{\tau_2}{2}\left(t^{-1} F_{\Lambda}(z) - t\, \bar{F}_{\Lambda}(\bar{z})\right)$$ D3-D5

$$\alpha = \sigma + \frac{\tau_2}{2}\left(t^{-1} W(z) - t\, \bar{W}(\bar{z})\right) + \frac{i\chi(\mathcal{X})}{24\pi}\log t$$ NS5

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$$\xi^{\Lambda}_{[i]}(x,t),\ \ \tilde{\xi}^{[i]}_{\Lambda}(x,t),\ \ \alpha^{[i]}(x,t)$$

- In the absence of NS5-branes, isometries of α are unbroken

→ $H^{[ij]}(\xi, \tilde{\xi}, \alpha) = H^{[ij]}(\xi, \tilde{\xi})$ reduce to symplectomorphisms

- When NS5-branes are present, $H^{[ij]}(\xi, \tilde{\xi}, \alpha)$ are genuine contact transformations

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Translations along $\mathcal{T}$ form a Heisenberg group with algebra: $[T_{\xi^\Lambda}, T_{\tilde{\xi}_\Sigma}] = k\delta^\Lambda_\Sigma$

NS5 charge

$\longrightarrow$ Cannot diagonalize T_{ξ^Λ} and $T_{\tilde{\xi}_\Sigma}$ simultaneously

Must choose a polarization!

- $H^{[ij]}(\xi,\tilde{\xi},\alpha)$ has a "non-abelian Fourier expansion":

$$H^{[ij]}(\xi,\tilde{\xi},\alpha) = \sum_{k\neq 0} \sum_{\mu\in(\Gamma_m/|k|)/\Gamma_m} \sum_{n\in\Gamma_m+\mu} H_{\mu,k}(\xi^\Lambda - n^\Lambda)\,\mathbf{E}\left[kn^\Lambda\tilde{\xi}_\Lambda - \frac{k}{2}\left(\tilde{\alpha}+\xi^\Lambda\tilde{\xi}_\Lambda\right)\right]$$

"Wave function"

[Ishikawa][Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline]

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"Wave function"

Suggestive relation with the topological string...

NS5-instantons in twistor space

Aim: Find the twistor space analogue of the fivebrane partition function

General arguments require:

$$H_k(\xi, \tilde{\xi}, \alpha) \sim e^{-i\pi k\alpha} \mathcal{Z}^{(k)}(\xi, \tilde{\xi})$$

NS5-instantons in twistor space

Aim: Find the twistor space analogue of the fivebrane partition function

"Exploratory strategy": Start from the D-instanton series in IIB and employ S-duality

$\longrightarrow$ (D5, NS5) form a doublet under $SL(2,\mathbb{Z})$

- Similar philosophy as was previously done for (F1,D1) + D(-1)

[Robles-Llana, Rocek, Saueressig, Theis, Vandoren]

NS5-instantons in twistor space

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- Deformations arise due to to D(-1)-D1-D3-D5 wrapping

$$\gamma = p^0 + p^a\omega_a - q_a\omega^a + q_0\omega_{\hat{\mathcal{X}}} \in H_{\text{even}}(\hat{\mathcal{X}},\mathbb{Z})$$

- "BPS ray" $\ell_\gamma = \{t:\ Z_\gamma(z^a)/t \in i\mathbb{R}^-\}$

- Across each ray, $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ are related by a symplectomorphism

[Kontsevich, Soibelman][Gaiotto, Moore, Neitzke] [Alexandrov, Saueressig, Pioline, Vandoren]

$$H_\gamma = \frac{i}{2(2\pi)^2}\Omega(\gamma)\mathrm{Li}_2\left[\sigma_\Theta(\gamma)\, e^{-2\pi i(q_\Lambda\xi^\Lambda - p^\Lambda\tilde{\xi}_\Lambda)}\right]$$

same QR as in the fivebrane partition function!

generalized DT-invariants

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[Kontsevich, Soibelman][Gaiotto, Moore, Neitzke] [Alexandrov, Saueressig, Pioline, Vandoren]

$$H_\gamma = \frac{i}{2(2\pi)^2}\,\tilde{\Omega}(\gamma)\sigma_\Theta(\gamma)\,e^{-2\pi i(q_\Lambda\xi^\Lambda - p^\Lambda\tilde{\xi}_\Lambda)}$$

$$\tilde{\Omega}(\gamma) = \sum_{d|\gamma}\frac{1}{d^2}\,\Omega(\gamma/d)$$

S-duality

- Data: $(H_\gamma, \ell_\gamma) \longrightarrow \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\hat{\gamma}}, \ell_{k,p,\hat{\gamma}})$

$$\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

- $\gcd(c, d) = 1$
- $(c, d) = (-k, p)/p^0$

k NS5-charge

p D5-charge

$p^0 = \gcd(k, p)$

- Reduced charge vector: $\hat{\gamma} = (p^a, \hat{q}_a, \hat{q}_0)$

S-duality

- Data: $(H_\gamma, \ell_\gamma) \longrightarrow \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\hat\gamma}, \ell_{k,p,\hat\gamma})$

- Key property: $\begin{pmatrix} \tilde\xi_0 \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde\xi_0 \\ \alpha \end{pmatrix} + \cdots$

$$\tilde\xi_0 = \tilde\zeta_0 + \frac{\tau_2}{2}\left(t^{-1} F_\Lambda(z) - t\,\bar F_\Lambda(\bar z)\right) \qquad \text{D5}$$

$$\alpha = \sigma + \frac{\tau_2}{2}\left(t^{-1} W(z) - t\,\bar W(\bar z)\right) + \frac{i\chi(\mathcal{X})}{24\pi}\log t \qquad \text{NS5}$$

- Main assumption: S-duality leaves the complex contact structure invariant
  - May be questionable, but works for D(-1)-(F1,D1)

[Alexandrov, Saueressig, Pioline, Vandoren][Alexandrov, Saueressig]

S-duality

- Data: $(H_\gamma, \ell_\gamma) \longrightarrow \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\hat\gamma}, \ell_{k,p,\hat\gamma})$

- After some trickery: $(\mathbf{E}[x] = e^{2\pi i x})$

$$H_{k,p,\hat\gamma} = \frac{i\sigma_\Theta(\gamma)}{8\pi^2}\,\tilde\Omega(\gamma)\,\mathbf{E}\left[-\frac{k}{2}\,S_\alpha + \frac{p^0(k\hat q_a(\xi^a - n^a) + p^0\hat q_0)}{k^2(\xi^0 - n^0)}\right]$$

$$S_\alpha \equiv \alpha + (\xi^\Lambda - 2n^\Lambda)\tilde\xi_\Lambda + 2\frac{N(\xi^a - n^a)}{\xi^0 - n^0}$$

$$(n^0, n^a) = (p, p^a)/k$$

$$N(\xi^a) \equiv \frac{1}{6}\kappa_{abc}\xi^a\xi^b\xi^c$$

S-duality

- Data: $(H_\gamma, \ell_\gamma) \longrightarrow \delta \cdot (H_\gamma, \ell_\gamma) = (H_{k,p,\hat{\gamma}}, \ell_{k,p,\hat{\gamma}})$

- After some trickery: $(\mathbf{E}[x] = e^{2\pi i x})$

$$H_{k,p,\hat{\gamma}} = \frac{i\sigma_\Theta(\gamma)}{8\pi^2}\,\tilde{\Omega}(\gamma)\,\mathbf{E}\left[-\frac{k}{2}\,S_\alpha + \frac{p^0(k\hat{q}_a(\xi^a - n^a) + p^0\hat{q}_0)}{k^2(\xi^0 - n^0)}\right]$$

$$S_\alpha \equiv \alpha + (\xi^\Lambda - 2n^\Lambda)\tilde{\xi}_\Lambda + 2\frac{N(\xi^a - n^a)}{\xi^0 - n^0} \qquad (n^0, n^a) = (p, p^a)/k$$

- Up to subtle phases, the set of data $(\ell_{k,p,\hat{\gamma}}, H_{k,p,\hat{\gamma}})$ is invariant under Heisenberg translations of $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$

- Assuming that $\tilde{\Omega}(\gamma)$ is invariant under "generalized spectral flow"

Find the analogue of the fivebrane partition function

→ Consider the formal sum:

$$H^{(k)}_{\mathrm{NS5}}(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) = \sum_{p, p^a, q_\Lambda} H_{k,p,\hat{\gamma}}(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$$

Find the analogue of the fivebrane partition function

→ Use Heisenberg invariance to rewrite:

[Pioline, D.P.][Bao, Kleinschmidt, Nilsson, D.P., Pioline]

$$H^{(k)}_{\mathrm{NS5}}(\xi,\tilde{\xi},\alpha) = \sum_{\mu\in(\Gamma_m/|k|)/\Gamma_m} \sum_{n\in\Gamma_m+\mu} H^{(k,\mu)}_{\mathrm{NS5}}(\xi^\Lambda - n^\Lambda)\, \mathbf{E}\left[kn^\Lambda(\tilde{\xi}_\Lambda - \phi_\Lambda) - \frac{k}{2}\,(\tilde{\alpha} + \xi^\Lambda\tilde{\xi}_\Lambda)\right]$$

"NS5 wave function"

- This is a non-Gaussian theta series

→ Twistor space analogue of the chiral fivebrane partition function!

Does this make sense?

Restrict to $k = 1$ $\longrightarrow$ Considerable simplification:

$$H^{(1,0)}_{\rm NS5}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi^0} + \pi A_{\Lambda\Sigma}\xi^\Lambda\xi^\Sigma} \sum_{\hat{q}_a, \hat{q}_0} \tilde{\Omega}(\gamma)\,(-1)^{\hat{q}_0}\, e^{2\pi i \hat{q}_a \xi^a (\xi^0)^{-1} + 2\pi i \hat{q}_0 (\xi^0)^{-1}}$$

- Now compare with (rank 1) D6-D2-D0 DT-partition function:

$$\mathcal{Z}_{\rm DT} = \sum_{Q_a, J} (-1)^{2J}\, N_{DT}(Q_a, 2J)\, e^{-2\lambda J + 2\pi i Q_a z^a}$$

D2-charge D0-charge

Restrict to $k = 1$ $\longrightarrow$ Considerable simplification:

$$H^{(1,0)}_{\mathrm{NS5}}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi^0} + \pi A_{\Lambda\Sigma}\xi^\Lambda \xi^\Sigma} \sum_{\hat{q}_a, \hat{q}_0} \tilde{\Omega}(\gamma)\, (-1)^{\hat{q}_0}\, e^{2\pi i \hat{q}_a \xi^a (\xi^0)^{-1} + 2\pi i \hat{q}_0 (\xi^0)^{-1}}$$

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- Under formal change of variables: $(\lambda, z^a) = (2\pi/(i\xi^0), \xi^a/\xi^0)$

$$(Q_a, 2J) = (\hat{q}_a + c_{2,a}/24, \hat{q}_0)$$

$$\boxed{\longrightarrow \quad H^{(1,0)}_{\mathrm{NS5}}(\xi^\Lambda) = e^{-2\pi i \frac{N(\xi^a)}{\xi^0} + \pi A_{\Lambda\Sigma}\xi^\Lambda \xi^\Sigma}\, \mathcal{Z}_{\mathrm{DT}}(\xi^\Lambda)}$$

Restrict to $k=1$ $\longrightarrow$ Considerable simplification:

$$H^{(1,0)}_{\rm NS5}(\xi^\Lambda)=e^{-2\pi i\frac{N(\xi^a)}{\xi^0}+\pi A_{\Lambda\Sigma}\xi^\Lambda\xi^\Sigma}\sum_{\hat q_a,\hat q_0}\tilde\Omega(\gamma)\,(-1)^{\hat q_0}\,e^{2\pi i\hat q_a\xi^a(\xi^0)^{-1}+2\pi i\hat q_0(\xi^0)^{-1}}$$

Correct sign factor reproduced from the quadratic refinement!

$$\mathcal{Z}_{\rm DT}=\sum_{Q_a,J}(-1)^{2J}\,N_{DT}(Q_a,2J)\,e^{-2\lambda J+2\pi i Q_a z^a}$$

- Under formal change of variables: $(\lambda,z^a)=(2\pi/(i\xi^0),\xi^a/\xi^0)$

$$(Q_a,2J)=(\hat q_a+c_{2,a}/24,\hat q_0)$$

$$\boxed{\longrightarrow\quad H^{(1,0)}_{\rm NS5}(\xi^\Lambda)=e^{-2\pi i\frac{N(\xi^a)}{\xi^0}+\pi A_{\Lambda\Sigma}\xi^\Lambda\xi^\Sigma}\mathcal{Z}_{\rm DT}(\xi^\Lambda)}$$

Actually, a more accurate statement is:

$$H^{(1,0)}_{\mathrm{NS5}}(\xi^{\Lambda}) \sim \Psi^{\mathrm{top}}_{\mathbb{R}}(\xi)$$

Topological string amplitude in the "real" (background-independent) polarization

[Witten][Maulik, Nekrasov, Okounkov, Pandharipande][Denef, Moore][Schwarz, Tang]

Actually, a more accurate statement is:

$$H^{(1,0)}_{\mathrm{NS5}}(\xi^\Lambda) \sim \Psi^{\mathrm{top}}_{\mathbb{R}}(\xi)$$

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

$$H^{(1)}_{\mathrm{NS5}}(\xi,\tilde{\xi},\alpha) = e^{-i\pi\alpha} \sum_{n\in\Gamma_m} \Psi^{\mathrm{top}}_{\mathbb{R}}(\xi^\Lambda - n^\Lambda)\, \mathbf{E}\left[n^\Lambda \tilde{\xi}_\Lambda - \frac{1}{2}\xi^\Lambda\tilde{\xi}_\Lambda\right]$$

Fits nicely with earlier speculations: [Dijkgraaf, Verlinde, Vonk][Nekrasov, Ooguri, Vafa][Kapustin]

Actually, a more accurate statement is:

$$H^{(1,0)}_{\mathrm{NS5}}(\xi^{\Lambda}) \sim \Psi^{\mathrm{top}}_{\mathbb{R}}(\xi)$$

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

$$H^{(1)}_{\mathrm{NS5}}(\xi, \tilde{\xi}, \alpha) \equiv e^{-i\pi\alpha}\ \mathcal{Z}^{(1)}(\xi, \tilde{\xi})$$

- Now reexamine the NS5-coupling:

$\longrightarrow$ $\mathcal{Z}^{(1)}$ and $e^{i\pi\alpha}$ transform as a sections of $\mathcal{L}^{\mathbb{C}}_{\Theta} \longrightarrow \mathcal{T}^{\mathbb{C}}$ Ok!

$\longrightarrow$ Both factors are separately trivial under $\mathcal{L} \longrightarrow \mathcal{M}_c$ Ok!

Follows from: $\alpha = \sigma + \frac{\tau_2}{2}\left(t^{-1}W(z) - t\,\bar{W}(\bar{z})\right) + \frac{i\chi(\mathcal{X})}{24\pi}\log t$

Actually, a more accurate statement is:

$$H_{\mathrm{NS5}}^{(1,0)}(\xi^\Lambda) \sim \Psi_{\mathbb{R}}^{\mathrm{top}}(\xi)$$

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

$$H_{\mathrm{NS5}}^{(1)}(\xi,\tilde{\xi},\alpha) \equiv e^{-i\pi\alpha}\ \mathcal{Z}^{(1)}(\xi,\tilde{\xi})$$

Coupling is well-defined!

Actually, a more accurate statement is:

$$H^{(1,0)}_{\mathrm{NS5}}(\xi^\Lambda) \sim \Psi^{\mathrm{top}}_{\mathbb{R}}(\xi)$$

The twistor space fivebrane partition function in type IIB is a non-Gaussian theta series built from the A-model top. string amplitude

$$H^{(1)}_{\mathrm{NS5}}(\xi, \tilde{\xi}, \alpha) \equiv e^{-i\pi\alpha}\ \mathcal{Z}^{(1)}(\xi, \tilde{\xi})$$

→ The contribution from $k > 1$ NS5-branes should be given by the generating function of rank $r = \gcd(k, p)$ DT-invariants

But what about the Gaussian type IIA partition function?

→ By mirror symmetry this is given by a theta series based on the B-model topological string amplitude

But what about the Gaussian type IIA partition function?

- Consistency check: project onto the base $\mathcal{M}$ via the Penrose transform

$$e^{-i\pi\sigma}\mathcal{Z}^{(1)}(\mathcal{N},\zeta^\Lambda,\tilde{\zeta}_\Lambda)\ \text{“}=\text{”}\ \sum_{n\in\Gamma_m}\oint_{\mathcal{C}}\frac{dt}{2\pi i t}e^{-i\pi\alpha}\mathcal{Z}^{(1)}(\xi,\tilde{\xi})$$

$$\sim e^{-i\pi\sigma}e^{f_1(z)}\sum_{n\in\Gamma_m}\left[(\zeta^\Lambda-n^\Lambda)(\Im\mathcal{N}_{\Lambda\Sigma})z^\Sigma\right]^{\frac{\chi}{24}-1}e^{-S^{\mathrm{IIA}}_{\mathrm{Gauss}}(\mathcal{N},\zeta,\tilde{\zeta})}$$

$\longrightarrow$ Saddle point approx. reproduces Gaussian partition function with insertion

$\longrightarrow$ Phase of the normalization factor determined:

$$\mathcal{F}\sim e^{f_1(z)}$$

e^{f_1}: holomorphic part of the B-model one-loop amplitude $F_1=\log\left[e^{f_1(z)+\bar{f}_1(\bar{z})}/\sqrt{M(z,\bar{z})}\right]$

Summary and Discussion

- Initial steps towards understanding NS5-brane instantons in type II/CY3

→ Proposal for the normalization factor of (non-)chiral partition function

→ Non-linear partition function in twistor space

→ Proposal for contact transformation encoding NS5-deformations

→ Explicit relation with topological string wave functions and DT-invariants

Summary and Discussion

- Crucial open problem: understand wall crossing

→ KS-GMN symplectomorphisms should be upgraded to contact transformations

→ What about the motivic KSWCF?

Here is a suggestive hint: consider the Heisenberg group elements

$$\gamma = (n^\Lambda, m_\Lambda) \qquad T_\gamma \;:\; \begin{aligned} \xi^\Lambda &\longrightarrow \xi^\Lambda + n^\Lambda \\ \tilde{\xi}_\Lambda &\longrightarrow \tilde{\xi}_\Lambda + m_\Lambda \\ \alpha &\longrightarrow \alpha - m_\Lambda \xi^\Lambda + n^\Lambda \tilde{\xi}_\Lambda \end{aligned}$$

Summary and Discussion

- Crucial open problem: understand wall crossing

→ KS-GMN symplectomorphisms should be upgraded to contact transformations

→ What about the motivic KSWCF?

Here is a suggestive hint: consider the Heisenberg group elements

When acting on sections of the form $F_k(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) \equiv F_k(\xi^\Lambda, \tilde{\xi}_\Lambda) e^{-i\pi k \alpha}$

$$T_\gamma T_{\gamma'} = q^{\frac{1}{2}\langle \gamma, \gamma' \rangle} T_{\gamma+\gamma'}$$

→ quantum torus with deformation parameter: $q^{1/2} = -e^{i\pi k}$

$\boxed{\text{Quantum deformation} \longleftrightarrow \text{NS5-brane charge}}$

Summary and Discussion

- Crucial open problem: understand wall crossing

→ KS-GMN symplectomorphisms should be upgraded to contact transformations

→ What about the motivic KSWCF?

Here is a suggestive hint: consider the Heisenberg group elements

When acting on sections of the form $F_k(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha) \equiv F_k(\xi^\Lambda, \tilde{\xi}_\Lambda) e^{-i\pi k \alpha}$

Is there a "generalized" classical limit $q^{1/2} \to -e^{i\pi k}$ where the quantum dilogarithm reduces to a contact transformation?