TOPICS IN WALL-CROSSING

Note Title 8/5/2010

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"WALL-CROSSING FROM SUSY GALAXIES" 1008.0030

"BOUND-STATE TMN WALLS" 1008.3555

GAIOTTO - M - NEITZKE

"4D WC + 3D FT" 0807.4723

"WC, HITCHIN, WKB" 0907.3987

"FRAMED BPS STATES" 1006.0146

WALL-CROSSING IN COUPLED 2D-4D ????.????

Review: PiTP LECTURES ON BPS STATES.... Homepage

I. INTRODUCTION

WALL-CROSSING IN $D=4$, $\mathcal{N}=2$ THEORIES HAS BEEN KNOWN SINCE THE WORK OF SEIBERG & WITTEN.

THE 4D PHENOMENA WAS INSPIRED BY THE DISCOVERY OF A SIMILAR PHENOMENA IN THEORIES WITH $D=2$ $(2,2)$ SUSY (F, C, I, V).

CECOTTI & VAFA WENT ON TO GIVE W.C.F. FOR THESE $D=2$ SYSTEMS, BUT ONLY IN THE PAST FEW YEARS HAVE WE FOUND QUANTITATIVE W.C.F. FOR $D=4$, $\mathcal{N}=2$ SYSTEMS.

THIS TALK: AN OVERVIEW OF A POINT OF VIEW I HAVE DEVELOPED WITH MY COLLABORATORS:

MAIN RESULTS OF THIS OVERVIEW:

1. NEW INDICES & NEW BPS STATES:
   PSC & FRAMED BPS STATES

2. HALO PICTURE OF WALL CROSSING AND A NEW DERIVATION OF THE KSWCF VIA "BPS GALAXIES"

3. LINE DEFECTS & FRAMED BPS STATES LEAD TO "MOTIVIC KSWCF" AND Q-DEFORMATION OF FUNCTIONS ON SEIBERG-WITTEN MODULI SPACES.

4. LINE DEFECTS $\Rightarrow$ DARBOUX COORDINATES $\Longrightarrow$ CONSTRUCTION OF HK METRICS ON $\mathcal{M}$

5. FOR "A_1-THEORIES" DARBOUX COORD'S COINCIDE WITH FOCK-GONCHAROV COORD'S

6. SURFACE DEFECTS $\Rightarrow$ GENERALIZED 2D-4D WCF.

OVERVIEW OF THE OVERVIEW

II. $\mathcal{N}=2$ REVIEW

(A.) BPS STATES & THEIR INDICES

(B.) SELF-DUAL ABELIAN GAUGE THEORY

(C.) MARGINAL STABILITY & THE WALL-CROSSING PROBLEM.

III. PRIMITIVE & SEMI-PRIMITIVE WCF

(A.) MULTI-CENTER SOLUTIONS

(B.) BOUNDSTATE RADIUS

(C.) BASIC WC-MECHANISM

(D.) AMS PARADOX

(E.) HALOS + HALO FOCK SPACE

(F.) SEMIPRIMITIVE WCF

IV. BPS GALAXIES & KSWCF

(A.) NEED TO GO BEYOND SEMI-PRIMITIVE

(B.) BPS GALAXIES

- ENTROPIC SUPPRESSION

(C.) FRAMED BPS STATES, $\overline{\underline{\Omega}}$

(D.) FRAMED BPS W.C.: THE KS-TRANSFORMATIONS

(E.) DERIVING THE KSWCF

(F.) RESOLVING AMS PARADOX: CONJUGATION WALLS & GENERALIZATION OF KSWCF

V. LINE DEFECTS & FRAMED BPS STATES

(A.) DEF. OF L_ζ

(B.) $\mathcal{H}_{L_\zeta} = \bigoplus_\gamma \mathcal{H}_{L_\zeta, \gamma}$

FRAMED BPS STATES

$E \geq -\mathrm{Re}(Z_\gamma/\zeta)$

(C.) GENERATING FUNCTION

HALOS AGAIN

MOTIVIC KSWCF

(D.) STRONG POSITIVITY,

CLUSTER ALGEBRAS,

AND TROPICAL LABELS

VI : HYPERKÄHLER GEOMETRY: COMPACTIFICATION TO THREE DIMENSIONS

(A.) $\mathcal{M}$ AS TORUS FIBRATION

(B.) DARBOUX COORDINATE EXPANSION OF $\langle L_\zeta \rangle$

(C.) PROPERTIES OF $\mathcal{Y}_\gamma$

(D.) CONSTRUCTING THE $\mathcal{Y}_\gamma$ VIA TBA

(E.) CONSTRUCTION OF HK METRICS

(F.) QUANTUM DEFORMATION OF $\mathcal{M}$

VII. M5 BRANES & HITCHIN SYSTEMS

(A.) $\mathcal{S}$ = (2,0) $\mathfrak{g}$-THEORIES / $C \times \mathbb{R}^{1,3}$

(B.) 3D COMPACT: $\mathcal{M}$ = HITCHIN MODULI

(C.) $\exists$ $\mathcal{L}$ SET OF LINE OPERATORS LABELED BY $\wp \subset C$

(D.) $\langle L_{\zeta}(\wp) \rangle$ = Tr Hol (FLAT GAUGE FLD)

(E.) FOR A_1 - HITCHIN SYSTEMS

- $\mathcal{Y}_\gamma$ = FOCK-GONCHAROV COORD'S
- $\exists$ ALGORITHM FOR COMPUTING
  - BPS INDICES $\Omega(\gamma)$
  - FRAMED INDICES $\overline{\underline{\Omega}}(\gamma)$

VIII. SURFACE DEFECTS & 2D/4D WCF

(A.) DEFINITION OF SURFACE DEFECTS
PRESERVED SUSY

(B.) EXAMPLES

(C.) IR DESCRIPTION :
GUKOV-WITTEN PARAMETERS
& TORSORS Γ_2.

(D.) INCLUDING LINE DEFECTS

(E.) FORMAL STRUCTURE OF
2D/4D WCF

(F.) 3D GEOMETRY: HHVB'S
$V \rightarrow \mathcal{M}$

(G.) EXAMPLES: HITCHIN SYSTEMS

II. $\mathcal{N}=2$ REVIEW

STUDY PARTICLE SPECTRUM OF THEORIES WITH $D=4, \mathcal{N}=2$:

(A.) SUSY ALGEBRA

$$\mathcal{S} = \mathcal{S}^0 \oplus \mathcal{S}^1$$

$$\mathcal{S}^0 = \mathrm{poin}(1,3) \oplus su(2)_R \oplus u(1)_R \oplus \mathbb{C}$$

$$\mathcal{S}^1 = \left[(2,1;2)_{+1} \oplus (1,2;2)_{-1} \right]_R$$

$$\left(Q_\alpha^A\right)^\dagger = \bar{Q}_{\dot{\alpha} A}$$

$$\{Q_\alpha{}^A, \bar{Q}_{\dot{\beta} B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \, \delta^A{}_B$$

$$\{Q_\alpha{}^A, Q_\beta{}^B\} = 2\epsilon_{\alpha\beta}\epsilon^{AB} \bar{Z}$$

(B.) A USEFUL INVOLUTION

FOR ANY PHASE ς DEFINE
AN <u>INVOLUTION</u> OF $SU(2,2|2)$
BY $\vec{x} \to -\vec{x}$ & $U(1)_R$ ROTATION BY ς

$$\mathcal{S}' = \mathcal{S}'^{,+} \oplus \mathcal{S}'^{,-}$$

$$R_\alpha{}^A := \xi^{-1} Q_\alpha{}^A + \xi\, \sigma^0_{\alpha\dot\beta}\, \bar{Q}^{\dot\beta A}$$

$$T_\alpha{}^A := \xi^{-1} Q_\alpha{}^A - \xi\, \sigma^0_{\alpha\dot\beta}\, \bar{Q}^{\dot\beta A}$$

$$\xi = \sqrt{\varsigma}$$

(C.) REPRESENTATIONS

$$\boxed{\text{MASSIVE PARTICLE} = \text{UNITARY IRREP. INDUCED FROM F.D. REP OF } \mathcal{S}^0_\ell \oplus \mathcal{S}^1}$$

$$\mathcal{S}^0_\ell = so(3) \oplus su(2)_R \oplus u(1)_R \oplus \mathbb{C}$$

$$\left(\mathcal{R}_1^1 + (\mathcal{R}_1^1)^+\right)^2 = 4\left(E + \mathrm{Re}(Z/\zeta)\right)$$

$$\Longrightarrow \boxed{E \geq -\mathrm{Re}(Z/\zeta)}$$

c.f. LINE DEFECTS BELOW

FOR $Z = e^{i\alpha}|Z|$ CHOOSE

$$\zeta = -e^{i\alpha} \Longrightarrow \boxed{M \geq |Z|}$$

$\mathcal{R}_\alpha{}^A$ & $\mathcal{T}_\alpha{}^A$ FORM (GRADED COMMUTING) CLIFFORD ALGEBRAS. EACH HAS IRREP

$$\rho_{hh} = (0;\tfrac{1}{2}) \oplus (\tfrac{1}{2};0)$$

AS REP OF $\mathfrak{s}^0_\ell = so(3) \oplus su(2)_R$

$M > |Z|$: LONG REPS

$$\rho_{hh} \otimes \rho_{hh} \otimes h$$

$M = |Z|$: SHORT REPS: $\mathcal{R}_\alpha{}^A = 0$,

$$\rho_{hh} \otimes h$$

h = ANY FIN. DIML. REP OF $so(3) \oplus su(2)_R$

PROTECTED SPIN CHARACTER

$$\mathrm{ch}(\rho) = \mathrm{Tr}_\rho\, x_1^{2J_3}\, x_2^{2I_3}$$

$$= \begin{cases} (x_1 + x_1^{-1} + x_2 + x_2^{-1})^2\, \mathrm{ch}(h) & \text{LONG} \\ (x_1 + x_1^{-1} + x_2 + x_2^{-1})\, \mathrm{ch}(h) & \text{SHORT} \end{cases}$$

$\therefore$

$$x_1 \frac{\partial}{\partial x_1}\left(\mathrm{Tr}_\rho\left(x_1^{2J_3}\, x_2^{2I_3}\right)\right)\Big|_{x_1 = -x_2 = y}$$

$$= \mathrm{Tr}\,(2J_3)(-1)^{2J_3}(-y)^{2(J_3+I_3)}$$

$$= (y - y^{-1})\,\mathrm{ch}(h)\Big|_{x_1 = -x_2 = y}$$

IS AN INDEX!

MODULI OF VACUA

1. $\mathcal{N}=2$ THEORIES HAVE MODULI SPACES OF VACUA

2. LOCALLY

$$\left\{ \begin{matrix} \text{VM} \\ \text{MODULI} \end{matrix} \right\} \times \left\{ \begin{matrix} \text{HM} \\ \text{MODULI} \end{matrix} \right\}$$

$\uparrow$

THIS TALK : $\mathcal{B}$

3. LOW ENERGY THEORY HAS UNBROKEN RANK r

SELFDUAL ABELIAN GAUGE THEORY

SELFDUAL ABELIAN GAUGE THEORY

1. LOCAL SYSTEM $\Gamma \to \mathcal{B}$ OF RANK $2r$ SYMPLECTIC LATTICES

(... MORE GENERALLY Γ IS POISSON:

$$0 \to \Gamma_{flavor} \longrightarrow \Gamma \longrightarrow \Gamma_{gauge} \to 0 \quad)$$

2. $V = \Gamma \otimes \mathbb{R}$ HAS COMPATIBLE POSITIVE COMPLEX STR. I

- $\mathbb{F} \in \Omega^2(M^4) \otimes V$
- $\mathbb{F} = (* \otimes I)\mathbb{F}$
- $d\mathbb{F} = 0$.

3. $\nexists$ LAGRANGIAN, BUT CHOICE OF DUALITY FRAME (α_I, β^I) FOR $\Gamma \Rightarrow$

- $\mathcal{I} \Longleftrightarrow \tau_{IJ}$
- $\frac{1}{4\pi}\int \mathrm{Im}\,\tau_{IJ}\, F^I * F^J + \mathrm{Re}\,\tau_{IJ}\, F^I F^J$

<u>Remark:</u> ALSO IN $D=6$ ABEL. TM ONCE ONE CHOOSES A LAG. DECOMPOSITION OF THE SPACE OF FIELDS $\exists$ AN ACTION PRINCIPLE (BELOV & MOORE)

CONSTRAINTS OF SUSY

$\mathcal{H}$ = 1- PARTICLE H.S.

$$= \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$$

RESTRICTION OF $\hat{Z}$ OPERATOR TO $\mathcal{H}_\gamma$ IS A SCALAR $\Longrightarrow$

$Z(\gamma)$: CENTRAL CHARGE <u>FUNCTION</u>

$\mathcal{N} = 2$ SUSY $\Longrightarrow$

(1) $Z(\gamma)$ DEPENDS ON VAC. MODULI BUT ONLY ON VM: $Z(\gamma; u)$

(2) $Z(\cdot\, ; u) \in \mathrm{Hom}(\Gamma, \mathbb{C})$

LIES IN A LAGRANGIAN SUBMFLD:

$$Z(\beta^I) = -a^I \qquad Z(\alpha_I) = a_{D,I} = \frac{\partial \mathcal{F}}{\partial a^I}$$

DEF. OF PROTECTED SPIN CHARACTER

$$\mathcal{H}^{BPS} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma^{BPS}$$ ← FINITE DIM'L

$$(y - y^{-1})\, \Omega(\gamma; u; y) := \mathrm{Tr}_{\mathcal{H}^{BPS}_{u,\gamma}} (2J_3)(-1)^{2J_3}(-y)^{2\mathcal{J}_3}$$

$$\boxed{\Omega(\gamma; u; y) = \mathrm{ch}\left(h_\gamma^{BPS}\right)\Big|_{x_1 = -x_2 = y}}$$

DEF: THE BPS INDEX IS

THE SPECIALIZATION TO $y = -1$:

$$\Omega(\gamma; u) := \Omega(\gamma; u; y)\Big|_{y=-1}$$

IMPORTANT REMARK

IN SUGRA THERE IS IN GENERAL NO $SU(2)_R$ SYMMETRY.
BUT BPS INDICES CAN STILL BE DEFINED, BECAUSE

$$\Omega(\gamma;u) = -\frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}}(2J_3)^2(-1)^{2J_3}$$

$$= \mathrm{Tr}_{h_{\gamma,u}}(-1)^{2J_3}$$

$$\Longrightarrow$$

$\exists$ SATISFACTORY PHYSICAL DERIVATION OF KSWCF FOR BPS INDICES IN BOTH FIELD THEORY & SUGRA, BUT ONLY FOR THEORIES WITH UNBROKEN $SU(2)_R$ IS THERE A PHYSICAL DERIVATION OF THE MOTIVIC W.C.F.

OPEN PROBLEM: FIND AN ALGORITHM TO COMPUTE THE BPS SPECTRUM

* NO EXAMPLES IN SUGRA
* $\exists$ FOR "A_1 – THEORIES IN CLASS $\mathcal{S}$"
* OTHERWISE UNKNOWN IN F.T., EXCEPT FOR SCATTERED EXAMPLES

FIRST STEP: HOW DOES THE SPECTRUM DEPEND ON MODULI?

- SPIN CHARACTER DEPENDS ON <u>BOTH</u> VM & HM
- PSC & INDEX ONLY DEPEND ON VM.

$\boxed{\text{BPS STATES CAN FORM BPS BOUNDSTATES}}$

FCIV, SW, ...

BINDING ENERGY

$$B.E. = |Z(\gamma_1+\gamma_2)| - |Z(\gamma_1)| - |Z(\gamma_2)| \leq 0$$

DECAYS ONLY ALONG

$$MS(\gamma_1,\gamma_2) := \{ u \,|\, Z(\gamma_1;u) \parallel Z(\gamma_2;u) \}$$

(B.E. < 0 NEC. NOT SUFF. !)

$\Omega(\gamma;u)$ ONLY JUMPS ACROSS $MS(\gamma_1,\gamma_2)$ WITH $\gamma_1+\gamma_2=\gamma$

$\boxed{\text{WCF: FIND } \Delta\Omega}$

III. PRIMITIVE & SEMI-PRIMITIVE WCF

(A.) MC. SOLN'S (DENEF)

DATA: $u = \lim_{\vec{x}\to\infty} u(\vec{x}) \in \mathcal{B}$

Collection $(\vec{x}_j, \gamma_j)$ $j = 1, \cdots, N$ s.t.

$$\forall i \quad \sum_{j\neq i} \frac{\langle \gamma_i, \gamma_j \rangle}{|\vec{x}_{ij}|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z_{\gamma_i}(u)\right)$$

$$e^{i\alpha} = \mathrm{Phase}\left(Z\left(\sum \gamma_i, u\right)\right)$$

Solution is a "BPS molecule" of dyonic BPS black holes.

(B.) $N=2$: BOUND STATE RADIUS:

$$R_{12} = \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 \bar{Z}_2)}$$

$$= \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle \frac{1}{\mathrm{Im}(e^{-i\alpha} Z_1)}$$

REMARKS:

(1) DENEF STABILITY:

$$\langle \gamma_1, \gamma_2 \rangle \mathrm{Im}(Z_1 \bar{Z}_2) > 0$$

NECESSARY BUT NOT SUFFICIENT

(2.) FOR $Z_2 \longrightarrow \infty$

- $|Z_1 + Z_2| - |Z_2| \to \mathrm{Re}\left(e^{-i\alpha} Z_1\right)$

$$= -\mathrm{Re}\left(Z_1/\zeta\right)$$

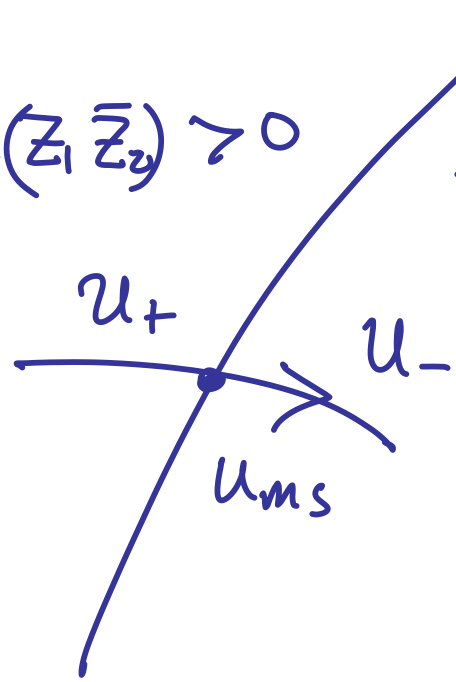
FOR $\zeta = -e^{+i\alpha} = -e^{i\alpha_2}$

- $2^{\underline{nd}}$ formula is valid provided $\langle \gamma_1, \gamma_2 \rangle$ Remains finite

(C.) BASIC WC MECHANISM:

(Denef + Moore)

$\langle\gamma_1,\gamma_2\rangle \mathrm{Im}(Z_1\bar{Z}_2) > 0$ $\qquad$ $\langle\gamma_1,\gamma_2\rangle \mathrm{Im}(Z_1\bar{Z}_2) < 0$



Radius $R_{12} \nearrow \infty \implies$

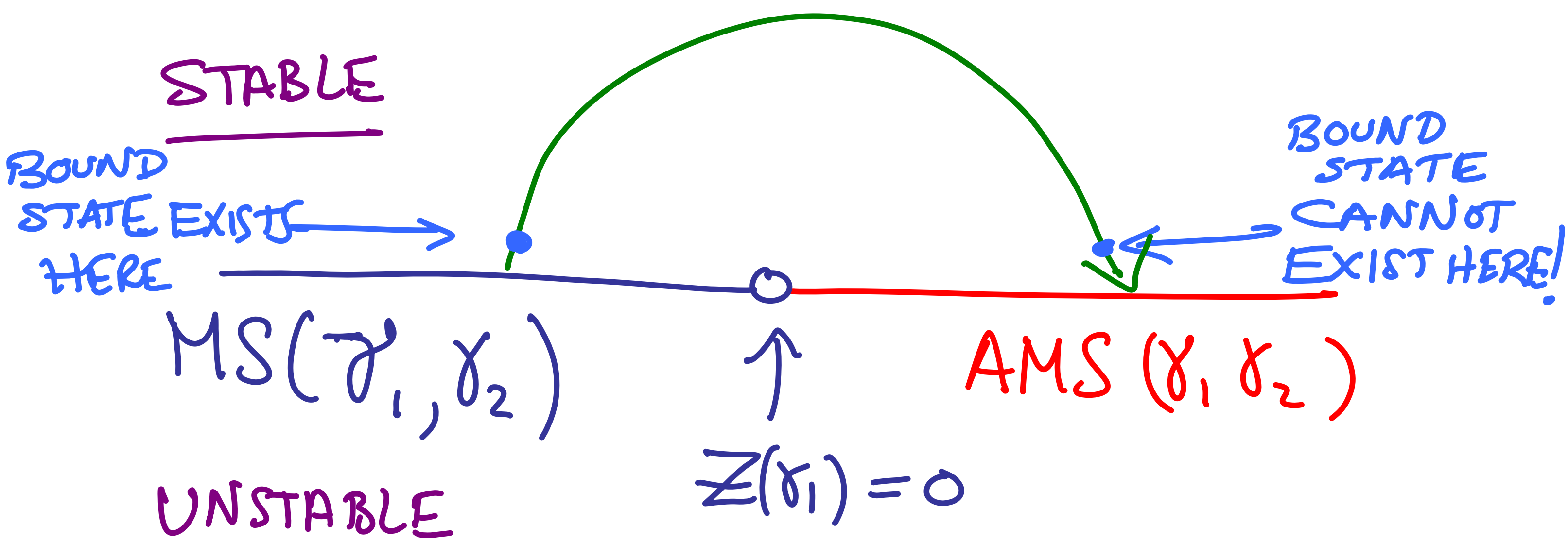
$$\Delta\Omega = ch\left(J_{\gamma_1,\gamma_2}\right)\Omega(\gamma_1,u_{ms};y)\,\Omega(\gamma_2,u_{ms};y)$$

γ_1, γ_2 PRIMITIVE

$$2J_{\gamma_1,\gamma_2}+1 = |\langle\gamma_1,\gamma_2\rangle|$$

(D.) AMS PARADOX

THIS PICTURE RAISES A PARADOX:



- $\nexists$ NO OTHER WALLS OF MS
- γ_1-PARTICLE IS STABLE IN A NBD $\mathcal{U}$ OF $Z(\gamma_1) = 0$.

RESOLUTION LATER ....

(E.) HALOS + HALO FOCK SPACES

ONE REASON THE PRIMITIVE WCF IS INCOMPLETE IS THAT

$$MS(\gamma_1, \gamma_2) = MS(N_1\gamma_1, N_2\gamma_2)$$

SO WHEN u CROSSES THE WALL LOTS OF OTHER STUFF DECAYS.

DEF: HALO CONFIG. IS A MC BOUNDSTATE WITH

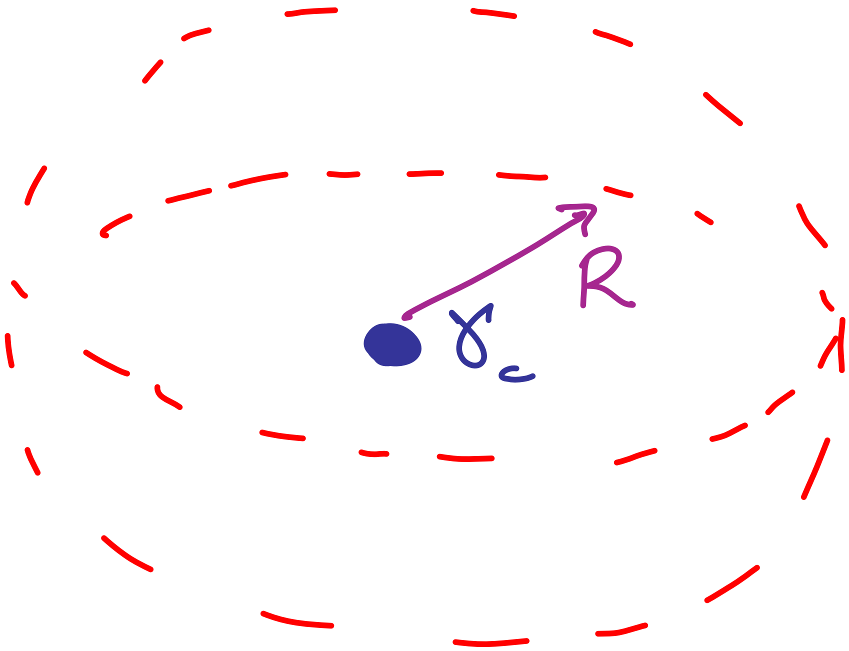
$$\vec{x} = 0 \ , \ \gamma_c$$

$$\vec{x}_i \ , \ \gamma_i = \lambda_i \gamma_h \qquad \lambda_i > 0$$

CONSTRAINTS $\Rightarrow$ ALL $\vec{x}_i$ SIT ON SPHERE:

$$R = \frac{1}{2}\langle \gamma_c, \gamma_h \rangle \frac{1}{\mathrm{Im}\left(e^{-i\alpha} Z_{\gamma_h}\right)}$$

AND NO OTHER CONSTRAINTS APPLY



HALO PARTICLES ARE MUTUALLY BPS $\Longrightarrow$ QUANTUM STATES ARE N-PARTICLE <u>FOCK SPACE</u> STATES.

ONE PARTICLE SPACE OF QUANTUM STATES: of a halo particle

$$\mathcal{H}^{BPS}_{u, \lambda\gamma_h} = \rho_{hh} \otimes h_{\lambda\gamma_h}$$

$\rho_{hh} \sim$ COM. MUST BE IN RADIAL SPIN $1/2$ STATE

$\Longrightarrow \cancel{\rho_{hh}}$ BUT ON THE OTHER HAND

THERE ARE LANDAU-LEVEL DEGENERACIES OF THE QUANTUM STATES WHICH ARE HENCE DRAWN FROM

$$W_{\lambda\gamma_h} = \left(J_{\gamma_c, \lambda\gamma_h}\right) \otimes \mathcal{H}_{\lambda\gamma_h}$$

$\mathbb{Z}_2$ - GRADING $-(-1)^{2J_3^h \lambda\gamma_h}$

HM's $\sim$ Halo fermions

VM's $\sim$ Halo bosons

The quantum states associated with halo configurations with core γ_c is:

$$\boxed{\bigoplus_{N\geq 0} q^N \mathcal{H}^{\text{HALO}}_{\gamma_c+N\gamma} = \mathcal{H}_{\gamma_c} \bigotimes_{n=1}^{\infty} \mathcal{F}\left[q^n W_{n\gamma_h} \right]}$$

$\mathcal{F}[W]$ = $\mathbb{Z}_2$-GRADED FOCK SPACE GENERATED BY $\mathbb{Z}_2$-GRADED V.S. W

(F.) SEMI-PRIMITIVE WCF.

NOW WRITE A WCF FOR THE HALO CONTRIBUTION TO THE INDEX:

Introduce group algebra for Γ

$$X_{\gamma_1} X_{\gamma_2} = X_{\gamma_1+\gamma_2}$$

$$G^{Halo}_{\gamma_c} = \sum_{N=0}^{\infty} \Omega^{Halo}_{\gamma_c}(N\gamma_h)\, X_{\gamma_c+N\gamma_h}$$

UNSTABLE $G^{Halo}_{\gamma_c} = X_{\gamma_c}$

STABLE :

$$G^{Halo}_{\gamma_c} = \prod_{n=1}^{\infty}\left(1-(-1)^{\langle n\gamma_h,\gamma_c\rangle n} X_{\gamma_h}\right)^{|\langle n\gamma_h,\gamma_c\rangle|\,\Omega(n\gamma_h;u_{ms})} \cdot X_{\gamma_c}$$

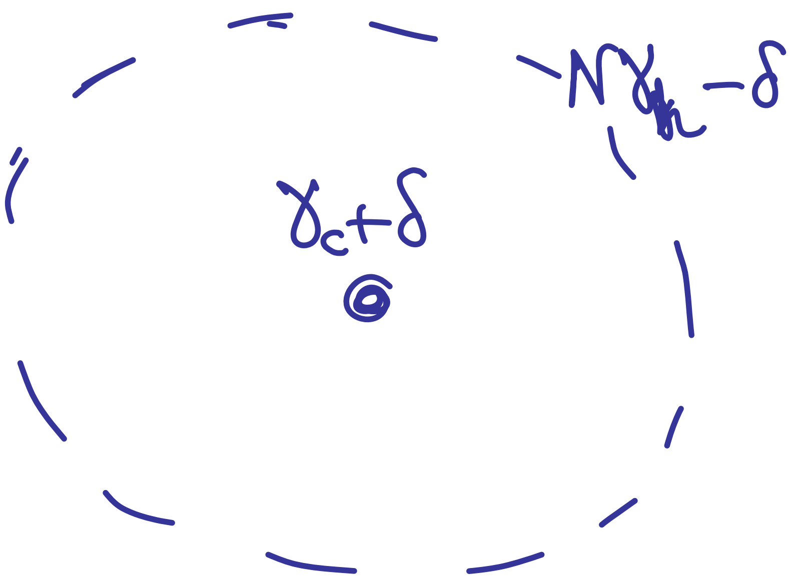
"SEMI-PRIMITIVE WCF"

IV. PHYSICAL DERIVATION OF KSWCF

(A.) MAIN PROBLEM W/ HALO PICTURE IS THAT

$$\Omega(\gamma_c + N\gamma) \neq \Omega(\gamma_c)\, \Omega^{\text{Halo}}_{\gamma_c}(N\gamma)$$

$\exists$ Mixing:



WITH OTHER MORE COMPLICATED BOUNDSTATES OF TOTAL CHARGE

$$\gamma_c + N\gamma$$

(B.) BPS GALAXIES

NEW IDEA : SUPPRESS MIXING BY TAKING LARGE CORE CHARGE

CHOOSE $U(1)$ SUBGROUP OF RANK r ABELIAN GAUGE GROUP

(el., mag.) CHARGES γ_0', γ_0,

$$\langle \gamma_0, \gamma_0' \rangle = 1$$

TAKE: $\gamma_c = \Lambda^2 \gamma_0 + \Lambda \gamma_0' + \delta$

$$\delta \in \Gamma_0^{\perp} := \{ \gamma \mid \langle \gamma, \gamma_0 \rangle = \langle \gamma, \gamma_0' \rangle = 0 \}$$

WE WILL TAKE $\Lambda \longrightarrow \infty$

WE DEFINE A "BPS GALAXY"

TO BE A M.C. SOLUTION

(OR ITS QUANTUM ANALOG)

WHERE ONE CENTER $\vec{x}=0$ HAS

CHARGE γ_c AND ALL OTHER

CENTERS $\vec{x}_j$ HAVE CHARGES

$\gamma_j \in \Gamma_0^{\perp}$.

KEY CLAIM: LET

$$\mathcal{G}(\gamma_c) = \left\{ \begin{array}{l} \text{ENSEMBLE OF ALL} \\ \text{BPS GALAXIES} \end{array} \right\}$$

FOR $\Lambda \to \infty$ THERE IS NO

QUANTUM MIXING BETWEEN

ENSEMBLES WITH $\gamma_c \neq \gamma_c'$

JUSTIFICATION

(1.) ENTROPIC SUPPRESSION:

BIG BLACK HOLES CAN'T FRAGMENT

cf. Maldacena-Michelson-Strominger

(2.) DISTANCE SUPPRESSION:

TUNNELING OVER LARGE DISTANCES IS EXPONENTIALLY SUPPRESSED

(C.) $\boxed{\text{FRAMED BPS INDEX}}$

$$\mathcal{H}_{\gamma_c}(\gamma_{orb}; u) := \mathcal{H}_{\gamma_c+\gamma_{orb}} / \mathcal{H}_{\gamma_c}$$

$$\overline{\underline{\Omega}}_{\gamma_c}(\gamma_{orb}; u) := \lim_{\Lambda \to \infty} \mathrm{Tr}_{\mathcal{H}_{\gamma_c}(\gamma_{orb}; u)} (-1)^{2J_3}$$

<u>Claim</u> These are well-defined and piecewise constant.

(D.) Framed BPS Wall-Crossing

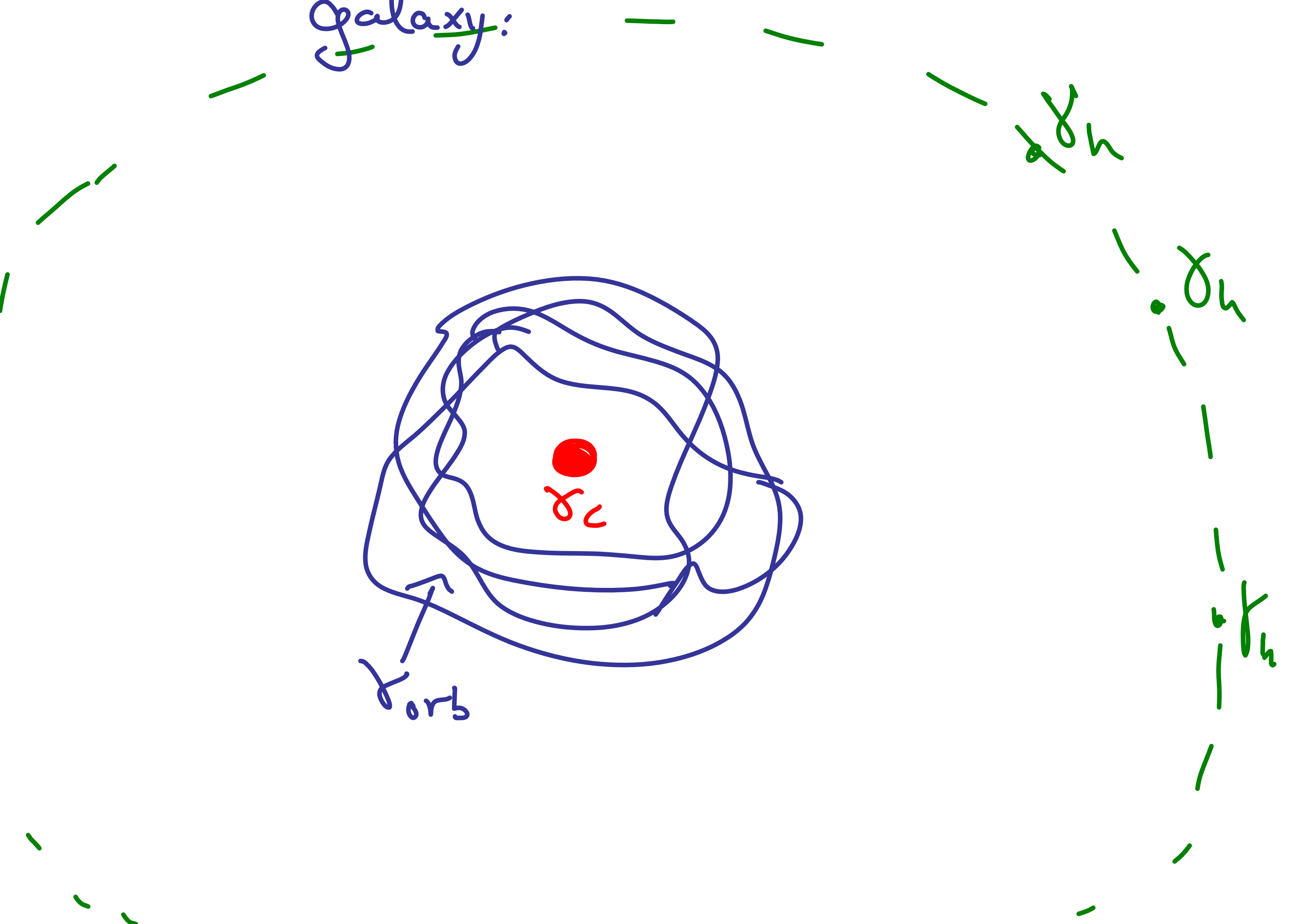
BPS Galaxies are complicated

But Wall-crossing is simple:

$$Z(\gamma_h; u) \parallel Z(\gamma_t; u)$$

$\Longrightarrow$ haloes $\quad \gamma_t := \gamma_c + \gamma_{orb}$

galaxy:



HAPPENS WHEN

$$Z(\gamma_h) \parallel Z(\gamma_c + \gamma_{orb})$$

$\Lambda \to \infty$:

$$W_{\gamma_h} = \{ u \mid Z(\gamma_h) \parallel Z(\gamma_0) \}$$

N.B. DEPENDENCE ON γ_{orb} DROPPED OUT!

NOW STUDY:

$$G_{\gamma_c} = \sum_{\gamma_{orb} \in \Gamma_0^{\perp}} \overline{\Omega}_{\gamma_c}(\gamma_{orb}; u) \; X_{\gamma_t} \cdot \left(X_{\gamma_0}^{-1} \, X_{\gamma_0'}^{2-1} \right)$$

HALOS $\Longrightarrow$

$$X_{\gamma_t} \to \prod_n \left(1 - (-1)^{\langle n\gamma_h, \gamma_t \rangle} X_{\gamma_h}^n \right)^{|\langle n\gamma_h, \gamma_t \rangle| \Omega(\gamma_h; u)} X_{\gamma_t}$$

Note $\gamma_h \in \Gamma_0^{\perp}$

$$\langle \gamma_h, \gamma_t \rangle = \langle \gamma_h, \delta + \gamma_{orb} \rangle$$

IS WELL-DEFINED FOR $\Lambda \to \infty$.

<u>BUT!</u> THE HALO FACTOR DEPENDS ON γ_{orb}.

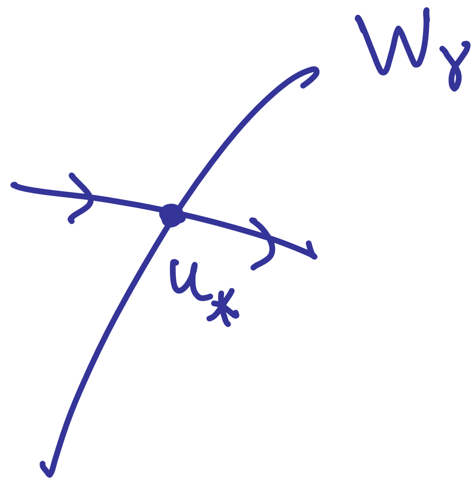
$\Longrightarrow$ INTRODUCE

$$D_\gamma X_{\gamma'} := \langle \gamma, \gamma' \rangle X_{\gamma'}$$

$$K_\gamma := \left(1 - (-1)^{D_\gamma} X_\gamma\right)^{D_\gamma}$$

Then across a BPS wall

$$G_{\gamma_c} \longrightarrow U_\gamma(u_*) \cdot G_{\gamma_c}$$

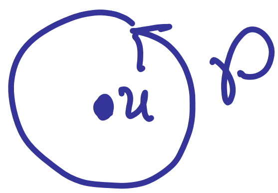


$$U_\gamma(u_*) = \prod_{n=1}^{\infty} K_{n\gamma}^{\Omega(n\gamma; u_*)}$$

IS A <u>DIFFL.</u> <u>OPERATOR</u>.

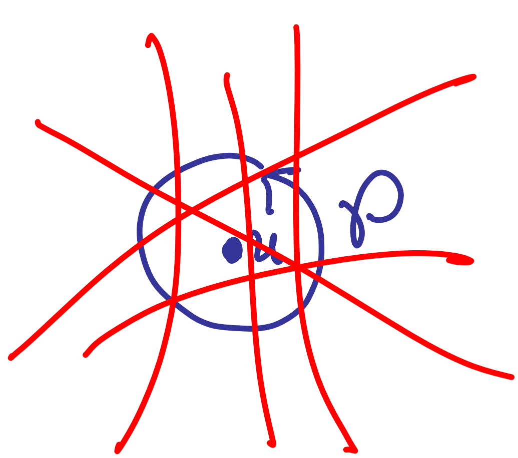
(E.) DERIVING KSWCF

CONSIDER A <u>NONSINGULAR</u> POINT IN MODULI SPACE AND A SMALL PATH $\wp$ NEARBY:



$G_{\gamma_c}(u)$ IS SINGLE-VALUED ON $\wp$

BUT THERE WILL TYPICALLY BE MANY WALLS W_γ



SO

$$G_{\gamma_c}(u) \longrightarrow \prod \mathcal{U}_{\gamma_i}(u_i) \cdot G_{\gamma_c}(u)$$

SINCE δ WAS ARBITRARY

$G_{\gamma_c}(u)$ IS ARBITRARY:

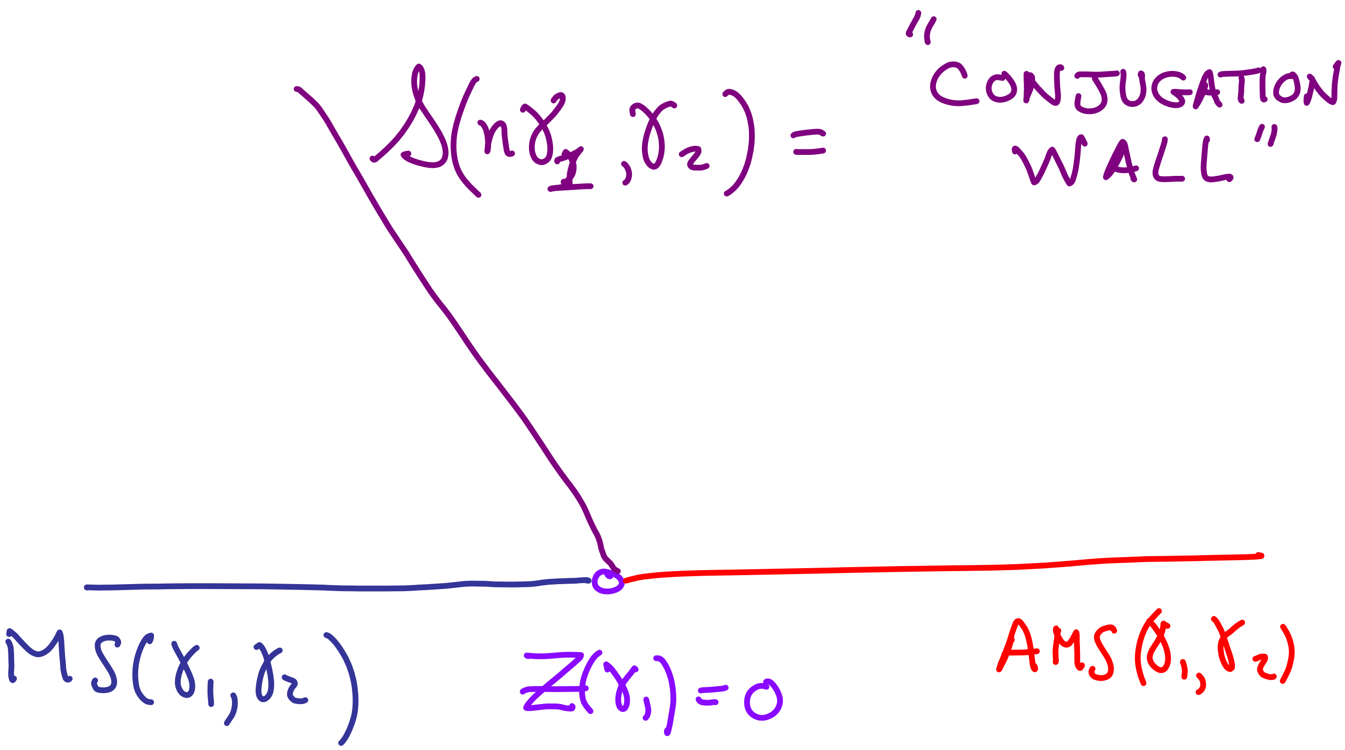
$$\boxed{\prod_i \mathcal{U}_{\gamma_i}(u_i) = 1}$$

OPERATOR EQUATION CONSTRAINS THE BPS DEGEN'S $\Omega(\gamma_i)$!

IT IS EASY TO SHOW THIS IMPLIES KSWCF.

(F.) AMS PARADOX

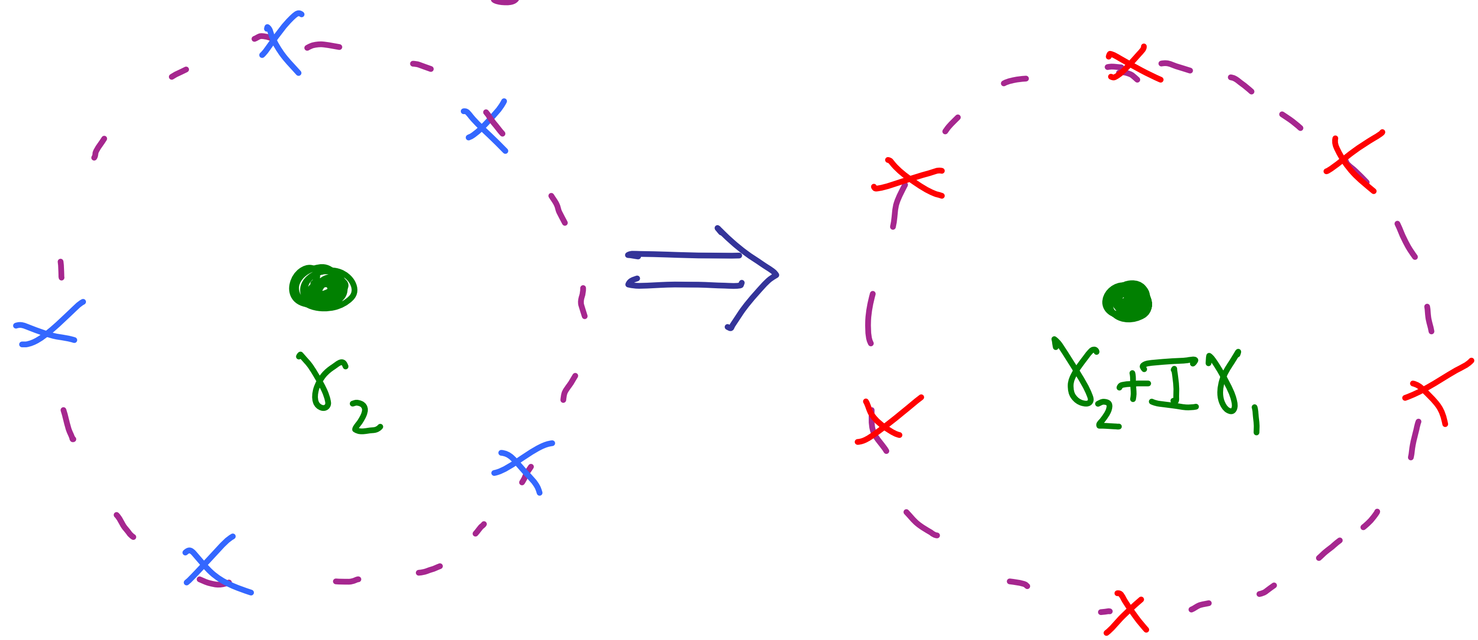
$\exists$ NEW WALL WHERE HALO DESCRIPTION CHANGES



$\mathcal{S}(n\gamma_1, \gamma_2)$ IS DEFINED IN TERMS OF ATTRACTOR FLOW TREES:

$$\{ u \mid \text{FLOW OF } n\gamma_1 + \gamma_2 \text{ CRASHES ON } Z(\gamma_1) = 0 \}$$

$n\ \gamma_1$-PARTICLES



$(I-n)\ (-\gamma_1)$-PARTICLES

I = MONODROMY:

$$\gamma_2 \longrightarrow \gamma_2 + I\,\gamma_1$$

$$I = |\langle \gamma_1, \gamma_2 \rangle| \sum_{k=1}^{\infty} k^2 \Omega(k\gamma) > 0 .$$

TWO CONSEQUENCES

1. IF THE ONLY MASSLESS POPULATED CHARGES AT $\mathcal{Z}(\gamma)$ ARE $\parallel$ to γ THEN:

$$P(q) = \prod_k (1-q^k)^{k\Omega(k\gamma)} \in \mathbb{Z}[q]$$

NOTA BENE: MASSLESS VM'S CONTRIBUTE $(1-q^k)^{-2k}$!

2. WHEN THERE ARE MASSLESS VM'S THERE ARE ALSO MASSLESS MONOPOLES !

V LINE DEFECTS

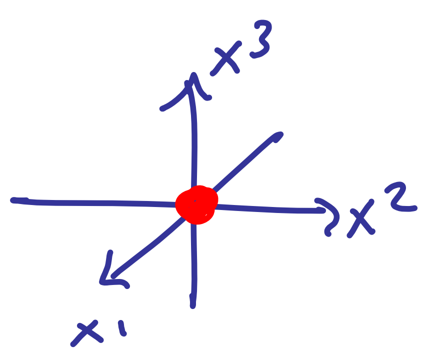
(A) NOW WE FOCUS ON FIELD THEORY BASED ON SCFML FXD PT $\mathcal{S}_*$

LINE DEFECT: NBD IS CONFORMAL TO $AdS_2 \times S^2$

DEFINE IT TO BE A CONFORMAL BDRY CONDITION OF $\mathcal{S}_*$ [KAPUSTIN]

CONSIDER A LINE DEFECT

$$\{\vec{x} = 0\} \times \mathbb{R}$$



DEF: WE SAY L IS OF TYPE ζ IF IT PRESERVES THE SUBALGEBRA FIXED BY g_ζ

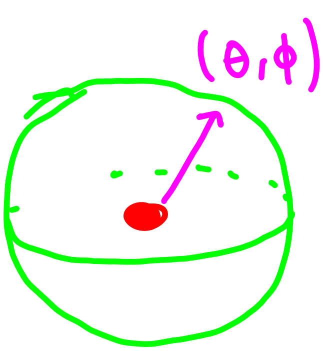
$$R_\alpha{}^A \sim Q_\alpha{}^A + \zeta\, \sigma^0_{\alpha\dot\beta}\, \bar{Q}^{\dot\beta A}$$

EXAMPLES

1.) WILSON

$$L_\zeta \sim P\exp \int_{\mathbb{R}\times\vec{0}} \left(\frac{\varphi}{2\zeta} - iA - \frac{\zeta}{2}\bar{\varphi} \right)$$

2.) 't HOOFT



$$F \sim p_\infty \sin\theta\, d\theta\, d\phi \qquad p \in t$$

$$\varphi/\zeta \sim \frac{p}{r} + \varphi_\infty/\zeta$$

(B.) FRAMED BPS STATES

HILBERT SPACE IS MODIFIED

$$\mathcal{H}_L = \bigoplus_{\gamma \in \Gamma_L} \mathcal{H}_{L,\gamma}$$

(IN GENERAL $\Gamma_L = \Gamma$ - TORSOR)

THINK OF $\mathcal{H}_{L,\gamma}$ AS SECTOR CREATED BY ∞-LY HEAVY DYON OF CHARGE γ $\Longrightarrow$

$$E \geq -\mathrm{Re}\left(Z_\gamma / \zeta\right)$$

DEF: FRAMED BPS STATES SATURATE THIS BOUND

DEF: THE FRAMED PROTECTED SPIN CHARACTER

$$\underline{\overline{\Omega}} := \mathrm{Tr}_{\mathcal{H}^{BPS}_{L_\zeta,\gamma,u}} (-1)^{2J_3} (-y)^{2\mathcal{J}_3}$$

$$\mathcal{J}_3 = J_3 + I_3$$

$\underline{\overline{\Omega}} \in \mathbb{Z}[y, y^{-1}]$ AND

$$\underline{\overline{\Omega}}(L, \gamma; y; \zeta; u)$$

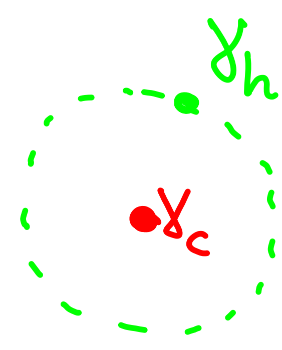
IS PIECEWISE CONSTANT IN ζ, u

(C.) FRAMED BPS WALL-CROSSING

NEAR BPS WALLS:

$$W(\gamma_h) = \{(u,\zeta) \mid Z_{\gamma_h}/\zeta < 0\}$$
$$\subset \mathcal{B} \times \mathbb{C}^*$$

SOME OF THE FRAMED BPS STATES ARE DESCRIBED BY HALOS:



$$r_{HALO} = \frac{\langle \gamma_h, \gamma_c \rangle}{2\,\mathrm{Im}(Z_{\gamma_h}/\zeta)}$$

(FOLLOWS FROM LIMIT OF BOUNDSTATE RADIUS IN THE LIMIT $Z_2 \to \infty$.)

CROSS $W(\gamma_h)$: CREATE/ANN. HALO FOCK SPACES

THE BPS WALLS $W(\gamma_h)$

DIVIDE $\mathcal{B} \times \mathbb{C}^*$ INTO CHAMBERS

<u>GENERATING FUNCTION:</u>

$$F(L,c) := \sum_{\gamma} \underline{\overline{\Omega}}(L;\gamma;y;c)\, X_\gamma$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}$$

	$W(\gamma_h)$
$\mathrm{Im}(Z_{\gamma_h}/\zeta) > 0$	$\mathrm{Im}(Z_{\gamma_h}/\zeta) < 0$
c_+	c_-
$F(L, c_+)$	$F(L, c_-)$

FOCK SPACE COMBINATORICS
ARE SUMMARIZED BY

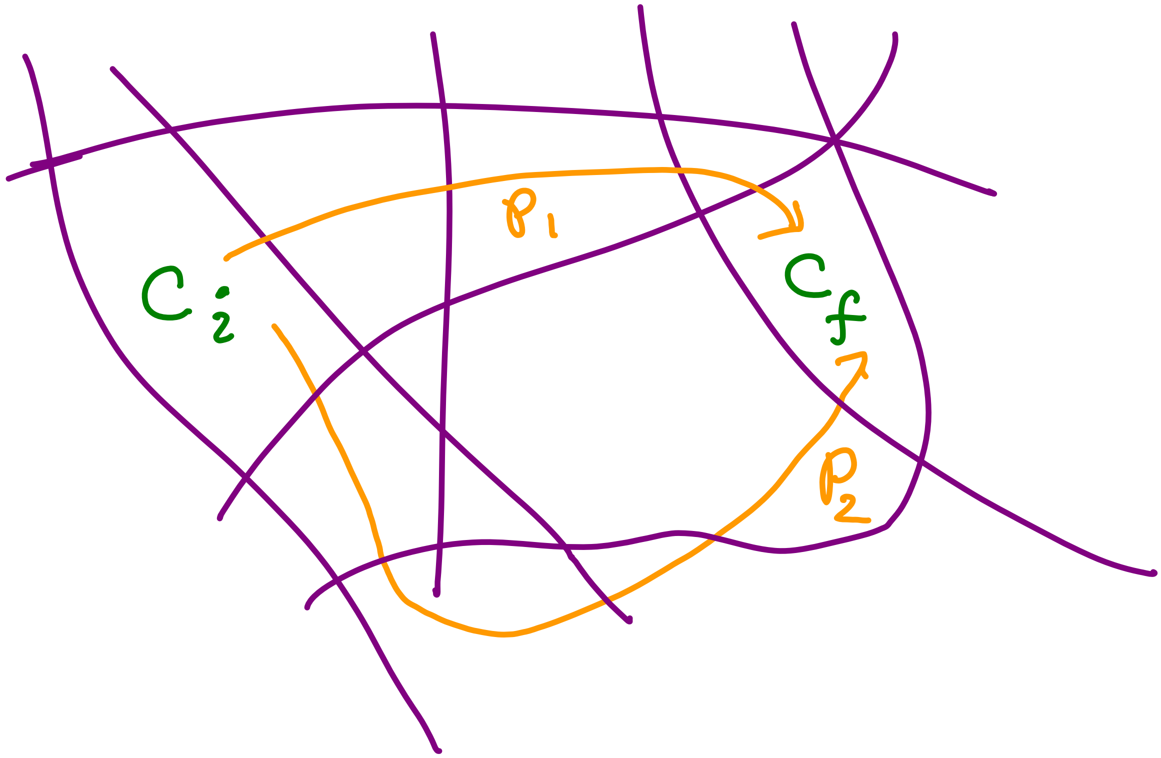
$$\boxed{F(L,c_+) = S_{\gamma_h} F(L,c_-) S_{\gamma_h}^{-1}}$$

$$S_{\gamma_h} = \prod_{m=-M_h}^{M_h} \Phi\left((-y)^m X_{\gamma_h}\right)^{a_{m,\gamma_h}}$$

$$\Phi(X) = \prod_{k=1}^{\infty}\left(1+y^{2k-1}X\right) = \begin{array}{c}\text{QUANTUM}\\ \text{DILOG}\end{array}$$

$$\underbrace{\mathrm{Tr}_{\mathcal{H}_{\gamma_h}}(-1)^{2J_3}(-y)^{2\mathcal{J}_3}}_{\text{PSC OF HALO PARTICLES}} = \sum_{-M_h}^{M_h} a_{m,\gamma_h}\, y^m$$

MOTIVIC KSWCF



$$F(L, c_f) = S(\wp_1)\, F(L, c_i)\, S(\wp_1)^{-1}$$
$$= S(\wp_2)\, F(L, c_i)\, S(\wp_2)^{-1}$$

MOTIVIC KSWCF: $S(\wp_1) = S(\wp_2)$

RELATED DISCUSSIONS:

DIMOFTE & GUKOV

CECOTTI & VAFA

DIMOFTE, GUKOV & SOIBELMAN

REMARK: $y = -1$

$$\hat{X}_{\gamma_1} \hat{X}_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \hat{X}_{\gamma_1 + \gamma_2}$$

$\hat{X}_\gamma$: COMMUTATIVE VARIABLES!

$$F(L, c_+) = \sum \overline{\underline{\Omega}}(L, \gamma) \hat{X}_\gamma$$

$$= K_{\gamma_h}^{-\Omega(\gamma_h)} F(L, c_-)$$

$$K_{\gamma_h}(\hat{X}_\gamma) := (1 - \hat{X}_{\gamma_h})^{\langle \gamma, \gamma_h \rangle} \hat{X}_\gamma$$

(D.) STRONG POSITIVITY

SURPRISING FACT: IN THE A_1- FIELD THEORIES

PSC = SPIN CHARACTER

(NONTRIVIAL STATEMENT ABOUT TOPOLOGY OF MONOPOLE MODULI SPACES.)

$\Rightarrow$ PURELY ALGEBRAIC CONSTRUCTION OF THE ALGEBRA OF LINE DEFECTS

DEF: A STRONGLY POSITIVE FORML LINE OPERATOR IS A COLLECTION $F(c)$ $c \in$ CHAMBERS

- $F(c) = \sum_{\gamma} P_{\gamma}^{c} X_{\gamma}$ FINITE SUM

P_{γ}^{c} = TRUE SPIN CHARACTER

$$= \sum_{n \geq 0} a_n \frac{y^n - y^{-n}}{y - y^{-1}} \qquad \boxed{a_n \in \mathbb{Z}_+}$$

- $F(c^+) = S_{\gamma_n} F(c^-) S_{\gamma_n}^{-1}$

CONJECTURE: FORMAL LINE OPERATORS COINCIDE WITH THE TRUE LINE OP'S

- CLUSTER ALGEBRAS
- CLUSTER VARIETIES
- TROPICAL LABELS

⋮

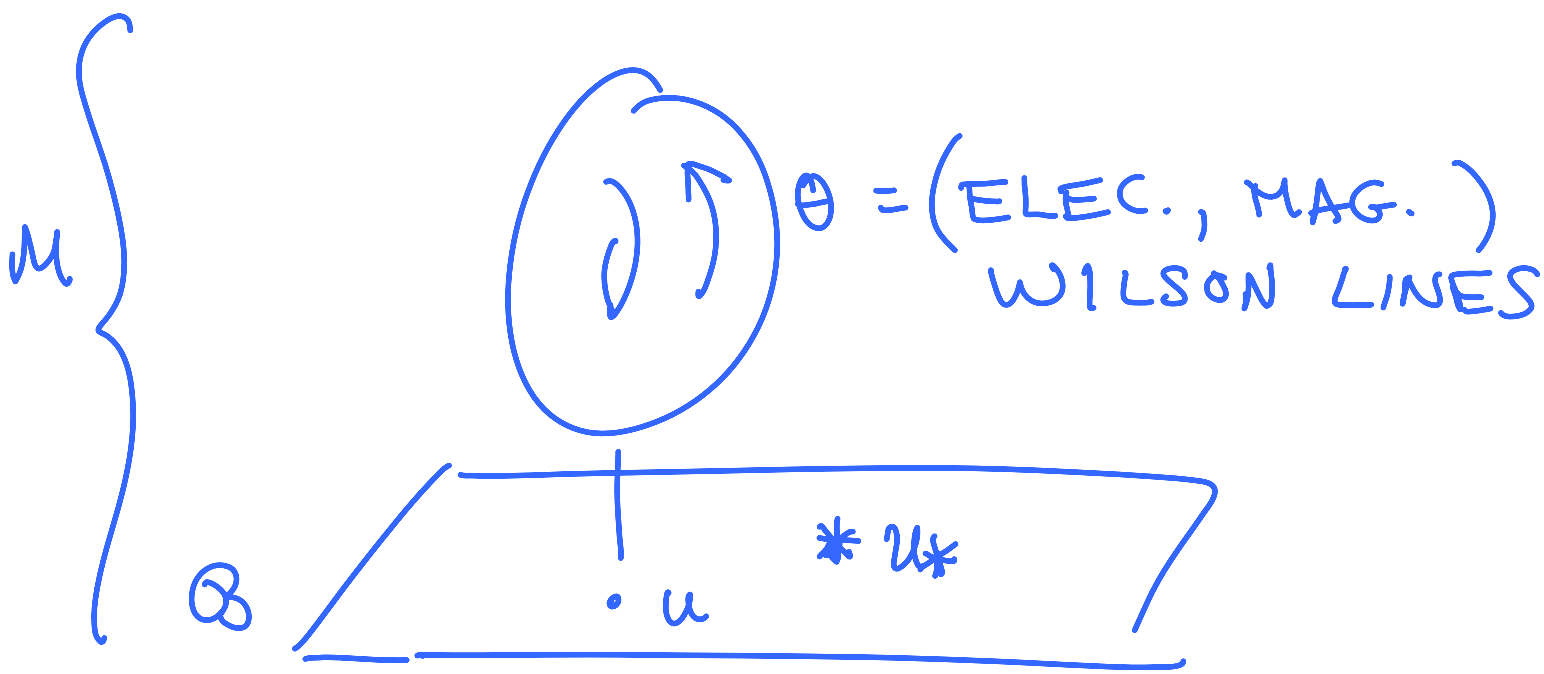
VI. HK GEOMETRY & COMPACTIFICATION TO 3DIM'S

(A.) PUT THEORY ON $\mathbb{R}^3 \times S^1_R$

LOW ENERGY EFFECTIVE THEORY:

σ-MODEL : $\mathbb{R}^3 \longrightarrow \mathcal{M}$

$\mathcal{M}$ IS A TORUS FIBRATION



SINGULAR ON DISCRIMINANT LOCUS WHERE POPULATED CHARGES γ ARE MASSLESS $Z_\gamma \to 0$.

(B.) "DARBOUX COORD'S"

- SUSY $\Rightarrow$ $\mathcal{M}$ IS HK

  $\Rightarrow$ FAMILY OF CPLX STRUCTURES $\mathcal{M}^{\zeta}$ $\zeta \in \mathbb{P}^1$

- WRAP L_ζ ON S^1 $\Rightarrow$

LOCAL OPERATOR IN 3D THEORY

$\langle L_\zeta \rangle$ IS HOLOMORPHIC ON $\mathcal{M}^{\zeta}$

- ON THE OTHER HAND

$$\langle L_\zeta \rangle = \mathrm{Tr}_{\mathcal{H}_{L_\zeta}} (-1)^{2J_3} e^{-2\pi R H} e^{i\theta \cdot Q} \sigma(Q)$$

$$\langle L_\zeta \rangle = \mathrm{Tr}_{\mathcal{H}_{L_\zeta}} (-1)^{2J_3} e^{-2\pi R H} e^{i\theta \cdot Q} \sigma(Q)$$

Q = CHARGE OPERATOR

$\theta \in \mathrm{Hom}(\Gamma, \mathbb{R}/\mathbb{Z})$ = B.C. @ ∞

ARE THE WILSON LINES

$\sigma(Q)$: QRIF: SELF-DUALITY!

POSIT AN EXPANSION

$$\langle L_\zeta \rangle = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma) \mathcal{Y}_\gamma$$

- UV DETAILS OF WHICH LINE OPERATOR CAPTURED BY $\overline{\underline{\Omega}}(L_\zeta, \gamma)$

- IR EFFECTIVE LINE OPERATOR CREATED BY ∞-ly HEAVY DYON OF CHARGE γ = $\mathcal{Y}_\gamma$ = "DARBOUX COORD'S"

(C.) PROPERTIES OF $\mathcal{Y}_\gamma$

(1.) $\mathcal{Y}_\gamma(\cdot, \zeta)$ HOLOMORPHIC ON $\mathcal{M}^\zeta$

(2.) HOLOMORPHIC IN $\zeta \in \mathbb{C}^*$

(3.) $R \to \infty$: TRACE ASYMPTOTES TO

$$\sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma)\, \underbrace{e^{\pi R \frac{Z_\gamma}{\zeta} + i \hat{\theta}\cdot\gamma + \pi R \zeta \bar{Z}_\gamma}}_{\mathcal{Y}_\gamma^{sf}}$$

SO:

$$\mathcal{Y}_\gamma \underset{R \to \infty}{\sim} \mathcal{Y}_\gamma^{sf}$$

LIKEWISE:

$$\mathcal{Y}_\gamma \underset{\zeta \to 0, \infty}{\sim} \mathcal{Y}_\gamma^{sf}$$

(4.) REALITY CONDITION

$$\overline{\mathcal{Y}_\gamma(\zeta)} = \mathcal{Y}_{-\gamma}(-1/\bar{\zeta})$$

(5.) RING STRUCTURE ON LINE OP'S $\Longrightarrow$

$$\mathcal{Y}_{\gamma_1}\mathcal{Y}_{\gamma_2} = (-1)^{\langle\gamma_1,\gamma_2\rangle}\mathcal{Y}_{\gamma_1+\gamma_2}$$

(6.) $\langle L_\zeta \rangle$ WELL-DEFINED IN UV & UNDERGOES NO PHASE TRANSITION $\Longrightarrow$

SMOOTH ON $\mathcal{M} \times \mathbb{C}^*$ $\Longrightarrow$

SMOOTH ACROSS BPS WALLS $W(\gamma_h)$

$\Longrightarrow$

$$F(L_\zeta) = \sum_\gamma \overline{\Omega}(L_\zeta, \gamma)\, \hat{x}_\gamma$$

$$\longrightarrow K_{\gamma_h}^{-\Omega(\gamma_h)} F(L_\zeta)$$

BUT

$$\langle L_\zeta \rangle = \sum_\gamma \overline{\Omega}(L_\zeta, \gamma)\, y_\gamma$$

$$\longrightarrow \langle L_\zeta \rangle$$

THEREFORE,

$$y_\gamma \longrightarrow K_\gamma^{\Omega(\gamma_h)}\, y_\gamma$$

THESE PROPERTIES DETERMINE THE y_γ UNIQUELY.

(D.) CONSTRUCTING THE $\mathcal{Y}_\gamma$

CONSTRUCT THESE FUNCTIONS VIA AN INTEGRAL EQUATION:

$$\boxed{\log \mathcal{Y}_\gamma = \log \mathcal{Y}_\gamma^{sf} + \sum_{\gamma'} \langle \gamma, \gamma' \rangle \Omega(\gamma') \, \mathbb{K}_{\gamma'} * \log\left(1 - \mathcal{Y}_{\gamma'}\right)}$$

$$\mathbb{K}_{\gamma'} * f = \int_{\ell_{\gamma'}} [d\zeta'] \, f(\zeta')$$

$$\ell_{\gamma'} = \{ \zeta \mid Z_{\gamma'}/\zeta < 0 \}$$

$$[d\zeta'] = \frac{-1}{4\pi i} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta}$$

FORMALLY IDENTICLE TO ZAMOLODCHIKOV'S TBA

(E.) CONSTRUCTING HK METRICS

USING STRUCTURE OF TORUS FIBRATION, AND THE ALGEBRA OF THE $\mathcal{Y}_\gamma$, WE CONSTRUCT THE

HOLOMORPHIC SYMPLECTIC FORM

$$\varpi_\zeta = \frac{-i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-$$

$$= \frac{1}{8\pi^2 R} C_{ij}\, d\log\mathcal{Y}_{\gamma^i} \wedge d\log\mathcal{Y}_{\gamma^j}$$

C_{ij} = inverse to $C^{ij} = \langle \gamma^i, \gamma^j \rangle$

RMK: QUANTUM DEFORMATION

$$F(L_\xi, c) = \sum_\gamma \overline{\underline{\Omega}}(L_\xi, \gamma, c; y) X_\gamma$$

DEFINES A NATURAL QUANTUM DEFORMATION OF FUNCTIONS ON $\mathcal{M}^\xi$

$$\{ y_\gamma, y_{\gamma'} \}_{\omega^\xi} = \langle \gamma, \gamma' \rangle y_\gamma y_{\gamma'}$$

THIS GENERALIZES A CONSTRUCTION OF FOCK-GONCHAROV AND TESCHNER

VII. M5-BRANES & HITCHIN SYSTEMS

(A.) 6D (2,0) THEORIES & CLASS $\mathcal{S}$

CONSIDER NONABELIAN $\mathfrak{g}$ - THEORY ON RIEMANN SURFACE C WITH PUNCTURES

BREAK $so(5)_R \longrightarrow so(2) \oplus so(3)$ AND PARTIALLY TWIST TO PRODUCE $d=4, \mathcal{N}=2$ THEORY ON $\mathbb{R}^{1,3}$

BOUNDARY COND'S AT PUNCTURES SPECIFIED BY BEHAVIOR OF BPS OPERATORS $\Longrightarrow$

$T[C, \mathfrak{g}, m]$

THIS IS THE CLASS $\mathcal{S}$.

(B.) CLASS $\mathcal{S}$ CONTAINS ALL THE FAMILIAR QUIVER THEORIES, AND GENERALIZED QUIVER THEORIES OF GAIOTTO.

WHEN COMPACTIFYING ON S^1 WE CAN IDENTIFY $\mathcal{M}$ WITH MODULI SPACE OF HITCHIN SYSTEMS ON C:

$$\begin{tikzcd}
 & 6D\ \mathfrak{g}\text{-THEORY}/C\times S^1\times\mathbb{R}^{1,2} \arrow[dl, "C"'] \arrow[dr, "S^1"] & \\
T[C,\mathfrak{g},m]/S^1\times\mathbb{R}^{1,2} \arrow[dr, "S^1"'] & & 5D\,SYM/C\times\mathbb{R}^{1,2} \arrow[dl, "C"] \\
 & \sigma\text{-MODEL } \mathbb{R}^{1,2}\rightarrow\mathcal{M} &
\end{tikzcd}$$

(C.) SW MODULI = HITCHIN MODULI

- REDUCTION OF 5D $\mathfrak{g}$-SYM ON C $\Rightarrow$ BPS EQS = HITCHIN EQS.

$$F + R^2 [\varphi, \bar{\varphi}] = 0$$

$$\bar{\partial}_A \varphi = 0$$

- SPECTRAL CURVE $\Sigma \subset T^*C$

$$\Sigma \longrightarrow C$$

= SEIBERG-WITTEN CURVE

- $d\lambda = dq \wedge dp$ ON T^*C

λ = SW DIFFL

$$\mathcal{M}^{\zeta}_{HITCHIN} \cong \mathcal{M}_{FLAT}(\mathfrak{g}_{\mathbb{C}}\text{-CONN})$$

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$$

FOR $\zeta \in \mathbb{C}^*$

(C.) A NATURAL SET OF LINE DEFECTS

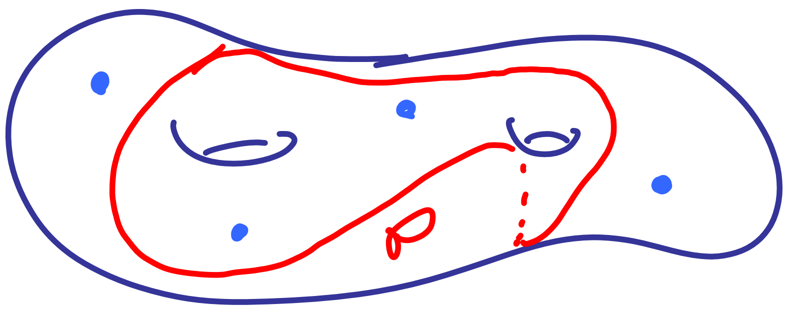
6D $\mathfrak{g}$ – THEORY HAS SUSY

<u>SURFACE OP'S</u> $\mathbb{S}(\mathcal{R}, \Sigma)$

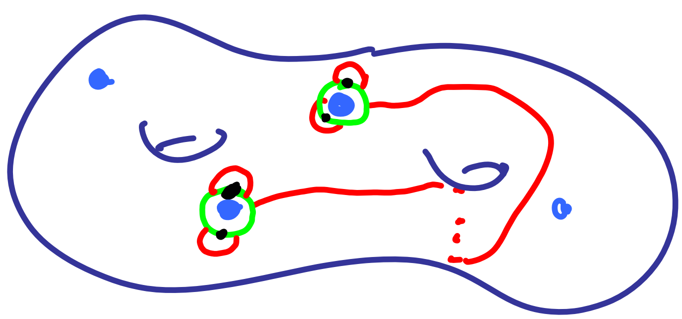
$\mathcal{R} \sim \mathfrak{g}$-REP, Σ = SURFACE

CHOOSE CLOSED CURVE $\wp \subset C$

C:



[MORE GENERALLY $\wp$ = LAMINATION



]

CHOOSE $\Sigma = S^1_R \times \wp$

6D SURFACE DEFECT $\mathbb{S}(\mathcal{R},\Sigma)$

$$\Longrightarrow$$

4D LINE DEFECT $L_\zeta(\mathcal{R},\wp)$

$$\langle \mathbb{S}(\mathcal{R},\Sigma)\rangle = \langle L_\zeta(\mathcal{R},\wp)\rangle$$

ON THE OTHER HAND

$$\langle \mathbb{S}(\mathcal{R},\Sigma)\rangle =$$

$$= \mathrm{Tr}_{\mathcal{R}}\left(P\exp\int_{\wp} \frac{\pi R\varphi}{\zeta} + A + \pi R\zeta\bar{\varphi} \right)$$

$$= \mathrm{Tr}_{\mathcal{R}}\left(\mathrm{Hol}_{\wp}(\mathcal{A})\right)$$

(D.) STRATEGY FOR COMPUTING FRAMED BPS DEG'S

- $\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$

  IS FLAT

- $Tr_{\mathcal{R}}(Hol_{\wp}\mathcal{A})$ IS HOLOMORPHIC ON $\mathcal{M}^{\zeta}$

- $\Longrightarrow$ NEED DARBOUX EXPANSION:

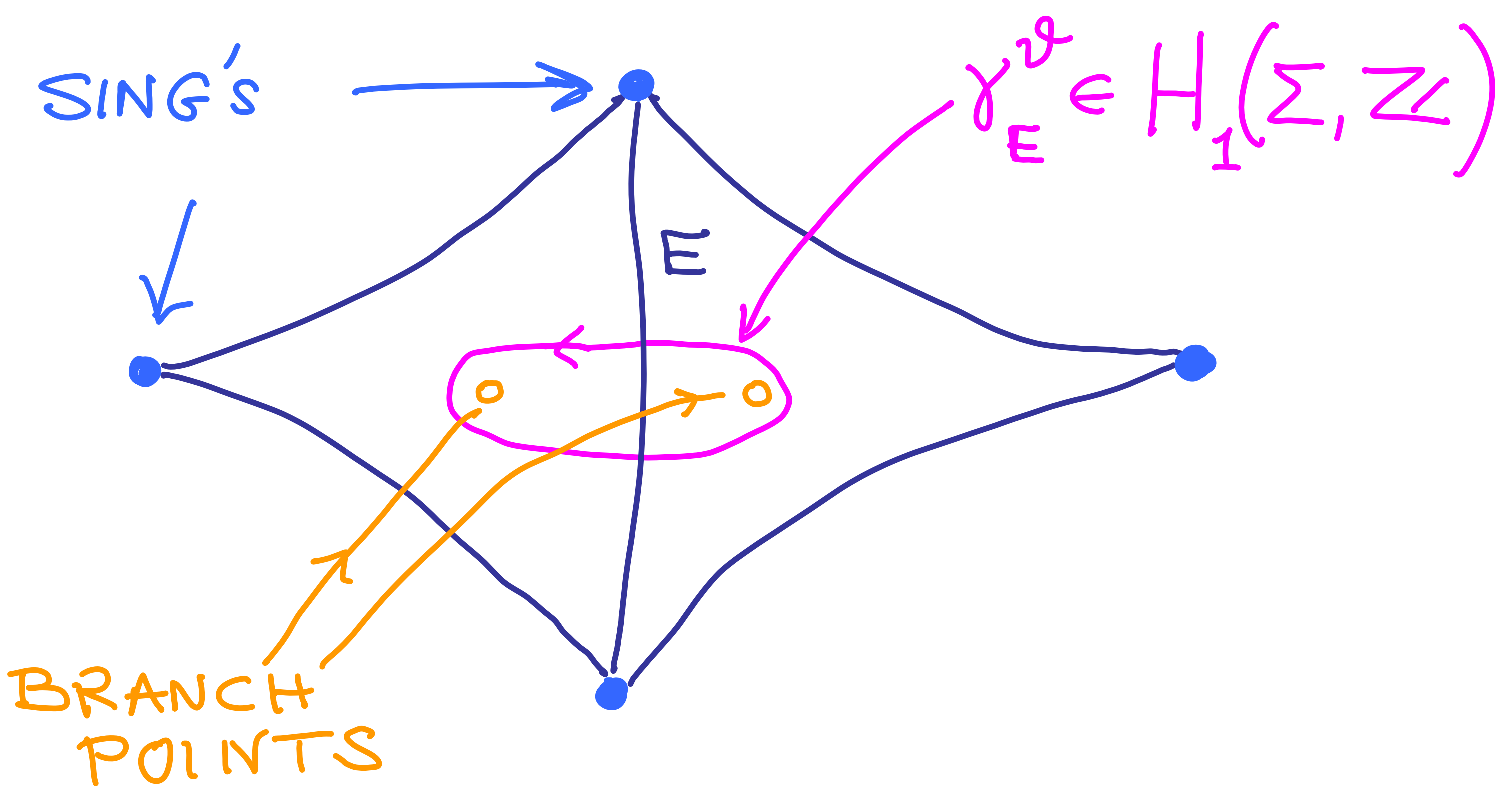
$$Tr_{\mathcal{R}}(Hol_{\wp}\mathcal{A}) = \sum_{\gamma} \overline{\underline{\Omega}}(L(\wp),\gamma)\,\mathcal{Y}_{\gamma}$$

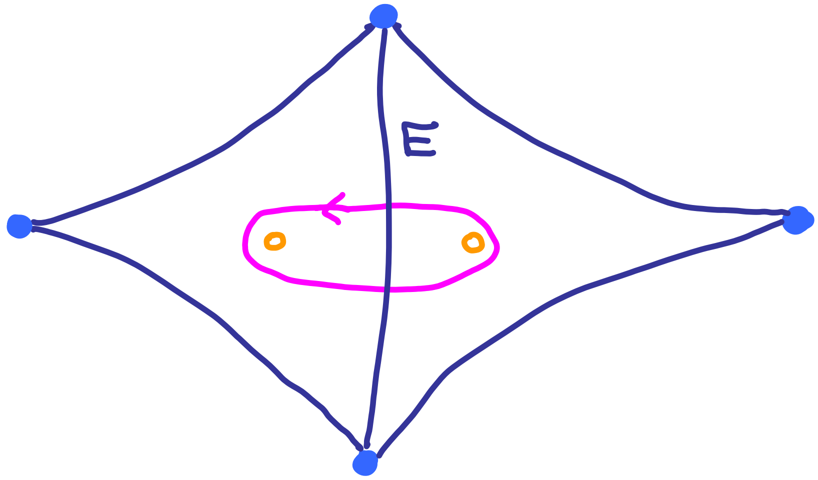
(E.) A_1 – THEORIES: HITCHIN SYSTEMS ON C WITH $\mathfrak{g} = sl(2)$

- $\Sigma \to C$ DOUBLE COVER

- FOR ANGLE ϑ DEFINE A DISTINGUISHED (DECORATED) <u>TRIANGULATION</u> OF C:

WKB CURVE: $\langle \lambda, \partial_t \rangle = e^{i\vartheta}$





$\mathcal{Y}_{\gamma_E}$ = FOCK-GONCHAROV COORDINATE X_E W.R.T. WKB TRIANGULATION

$\Rightarrow$ $\exists$ ALGORITHMS FOR COMPUTING $\Omega(\gamma)$ AND $\underline{\overline{\Omega}}(\gamma)$ PURELY IN TERMS OF THE COMBINATORICS OF THE TRIANGULATION.

(ONLY EXAMPLE WE HAVE WHERE BPS SPECTRUM IS COMPUTABLE BY AN ALGORITHM)

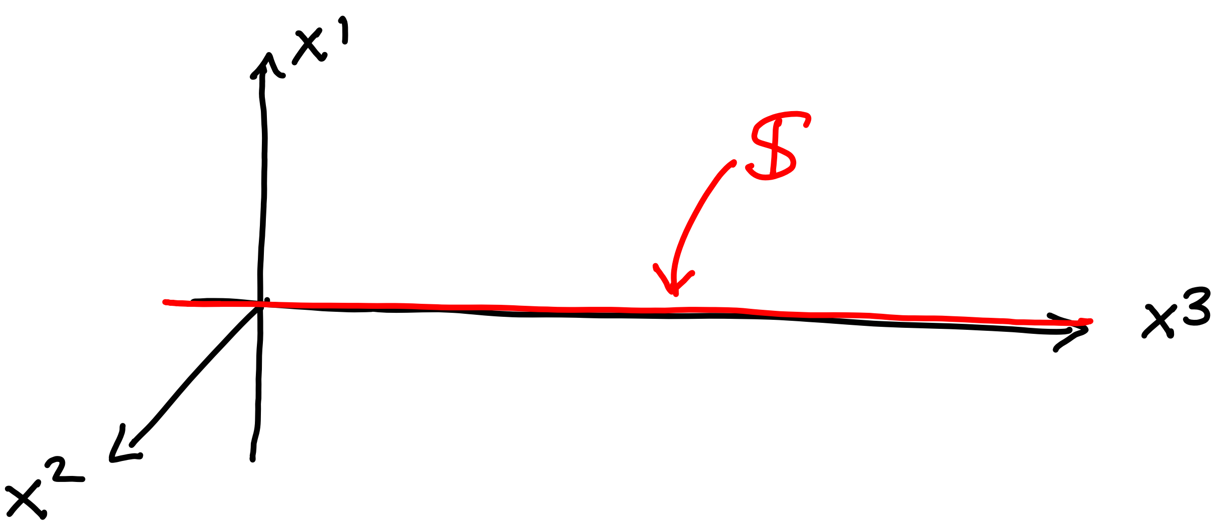
VIII. SURFACE DEFECTS & 2D/4D WCF

[MOTIVATION: PAPER BY GAIOTTO, 0911.1316]

(A.) UV DEFINITION

$\mathbb{S}$ SUPPORTED ON $S \subset M^4$

WE TAKE: $x^1 = x^2 = 0$ IN $\mathbb{R}^{1,3}$



AS BEFORE, DEFINE THE DEFECT BY SCFML BDRY COND. FOR UV THEORY $\mathcal{S}_*$ ON $AdS_3 \times S^1$

- PRESERVE SUPERALGEBRA

EVEN: $so(2,2) \oplus so(2)_{12} \oplus \left(U(1) \oplus U(1)\right)_R$

ODD: $\left\{ Q_\alpha{}^A \, , \, \bar{Q}_{\dot\alpha A} \right\}_{\alpha = A}$

- POINCARE' SUBALGEBRA

$$\{ P_0, P_3, M_{12}, F_A, F_V \}$$

$$\oplus$$

$$\{ Q_1{}^1, Q_2{}^2, \bar{Q}_{\dot 1 1}, \bar{Q}_{\dot 2 2} \}$$

IS ISOMORPHIC TO D=2 (2,2) SUSY

(<u>WITHOUT</u> SETTING $P_1 = P_2 = 0$!!)

$\begin{pmatrix} N=2 \\ VM \end{pmatrix} \supset$ D=2 (2,2) TWISTED CHIRALMULTIPLET

$$Y = \varphi + \theta \cdot \lambda + \theta^2 \left(F_{03} - i \left(F_{12} - D_{12} \right) \right)$$

(B.) EXAMPLES

1

- CHOOSE D=2 (2,2) THEORY T_{2d} ON S WITH G-FLAVOR SYMMETRY
- CHOOSE D=4 $\mathcal{N}=2$ THEORY T_{4d} WITH G-GAUGE SYMMETRY
- USING RESTRICTION OF TWISTED CHIRAL MULTIPLET, GAUGE THE G-FLAVOR SYMMETRY.

2

- USING D=4 THEORY T_{4d} WITH G-GAUGE SYMMETRY, REDUCE THE STRUCTURE GROUP ALONG $S \subset M_4$ TO $T \subset G$ AND INTRODUCE:

$$S = \int_{M_4} d^4x\, d^4\theta\, \mathcal{F} + \int_S dx^0 dx^3 d^2\tilde{\theta}\, W(\Upsilon) + \text{c.c.}$$

$$\Upsilon \in \mathfrak{t} = \mathrm{Lie}(T)$$

(C.) IR DESCRIPTION

- ASSUME $\mathfrak{S}$ HAS A FINITE SET OF MASSIVE VACUA $i \in \mathcal{V}$

- e.g. T_{2d} = L.G. MODEL WITH MORSE CRITICAL POINTS

- VACUA FOR THE 2D-4D SYSTEM ARE DESCRIBED BY (u, i) AND TOGETHER FORM A FINITE COVER:

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{B}_{\mathfrak{S}} \\ & & \downarrow \\ & & \mathcal{B} \end{array}$$

IR LAGRANGIAN

- 4D: UV $\longrightarrow$ IR

$$\mathcal{F} \longrightarrow \mathcal{F}^{eff}(V^I = a^I + \cdots)$$

4D ABELIAN VM'S
IN SOME DUALITY FRAME

- 2D+4D: UV $\longrightarrow$ IR

$$S^{eff} = \int d^4x\, d^4\theta\, \mathcal{F}^{eff}(V^I) + \int dx^0 dx^3 d^2\tilde{\vartheta}\, \mathcal{W}^{eff}(\Upsilon^I)$$

$\mathcal{F}^{eff}$ DEPENDS ON $u \in \mathcal{B}$

$\mathcal{W}^{eff}$ DEPENDS ON $(u, i) \in \mathcal{B}_{\mathbb{S}}$

$\mathcal{F}^{eff}, \mathcal{W}^{eff}$ ONLY LOCALLY DEFINED

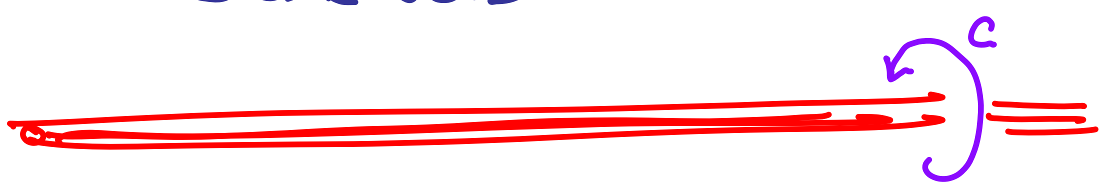
(D.) GUKOV – WITTEN PARAMETERS

$$t_I = \frac{\partial W^{eff}}{\partial a^I} = \eta_I + \tau_{IJ}\,\alpha^J$$

PHYSICAL INTERPRETATION:

SOLENOID

FLUX $= \gamma_i$ FOR VAC $= i$



$$\mathbb{F} = dA = 0$$

BUT $\oint_c A \in \bar{V} = \Gamma \otimes \mathbb{R}$

CHOOSE DUALITY FRAME

$$\oint_c A = \begin{pmatrix} \eta_I \\ \alpha^I \end{pmatrix}$$

N.B. AHARONOV–BOHM PHASE OF A PROBE PARTICLE

$$\gamma_t = \begin{pmatrix} p_t \\ q_t \end{pmatrix}$$

IS $\exp 2\pi i \left(p_t^I \eta_I - q_{t\,I} \alpha^I \right)$

- GAUGE TMNS
- MONODROMY $\Longrightarrow$ SHIFTS

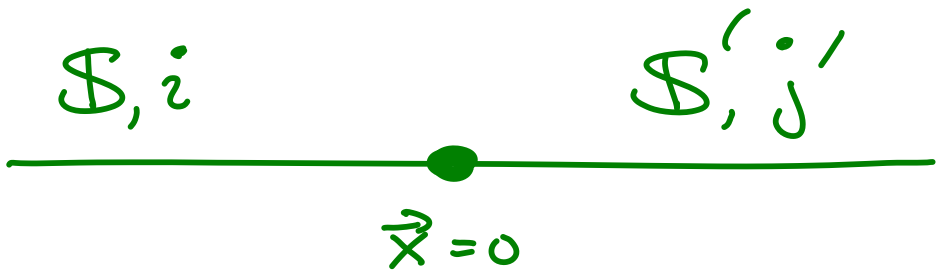
$$\begin{pmatrix} \eta \\ \alpha \end{pmatrix} \longrightarrow \begin{pmatrix} \eta \\ \alpha \end{pmatrix} + \gamma \qquad \gamma \in \Gamma$$

$\Longrightarrow$ G-W PARAMETERS LIVE IN A Γ-TORSOR: Γ_i

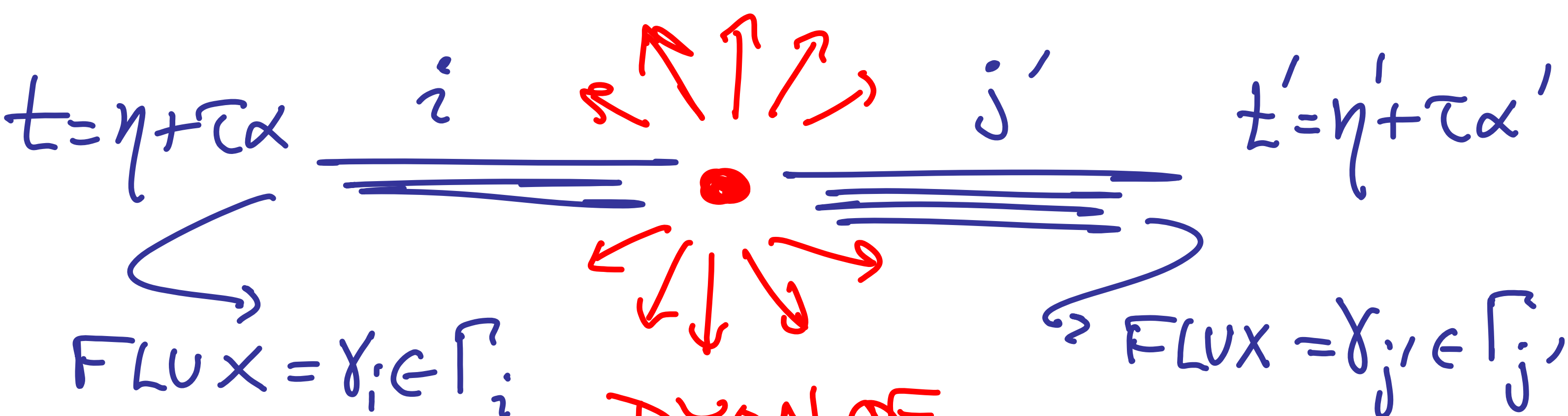
γ_i = FLUX IN SOLENOID

(E.) INCLUDING LINE DEFECTS

LINE DEFECTS = DOMAIN WALLS



IR: DYON EMBEDDED IN A SOLENOID:



DYON OF CHARGE $\gamma_i - \gamma_{j'} + \gamma \in \Gamma_{ij'}$

DYON CHARGE LIVES IN Γ-TORSOR $\Gamma_{ij'}$

(F.) BPS DEGENERACIES

- $\Omega(\gamma; u)$ AS BEFORE

- $\mu(\gamma_{ij})$: # 2D SOLITONS BETWEEN VAC i & j OF 4D CHARGE $\gamma_{ij} \in \Gamma_{ij}$

- $\omega(\gamma, \gamma_i)$: # 2D PARTICLES IN VACUUM i OF 4D CHARGE γ

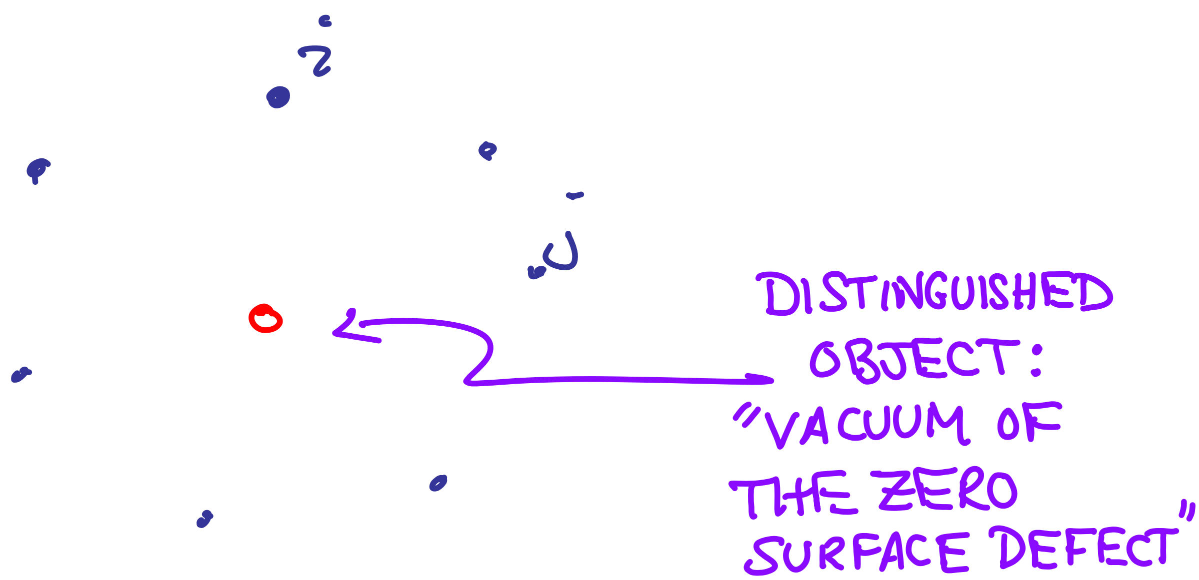
$\Rightarrow$ $\omega(\gamma, \gamma_i)$ IS SUBTLE TO DEFINE BECAUSE THESE STATES ARE AT THRESHHOLD

DEPENDS AFFINE-LINEARLY ON FLUX γ_i

(G.) FORMAL STATEMENT OF 2D4D WCF

4 PIECES OF DATA

(1.) GROUPOID OF VACUA $\mathbb{V}$:



MORPHISMS: $i \xrightarrow{\gamma} j$

DRAW AN ARROW FOR EACH $\gamma \in \Gamma$ TO GET $Hom(i,j) = \Gamma$-TORSOR.

$Hom(0,0) = \Gamma$, $Hom(i,0) = \Gamma_i$, ...

COMPOSITION OF MORPHISMS

$$\begin{tikzcd}
 & j \arrow[dr, "\gamma'"] & \\
i \arrow[ur, "\gamma"] \arrow[rr, "\gamma+\gamma'"'] & & k
\end{tikzcd}$$

etc.

LET $a \in \Gamma, \Gamma_i, \Gamma_{ij}$

DENOTE COMPOSITION $a+b$
WHEN DEFINED

2. CENTRAL CHARGE $Z \in \mathrm{Hom}(\mathbb{V}, \mathbb{C})$

$$Z(a) + Z(b) = Z(a+b)$$

WHEN $a+b$ IS DEFINED

3. BPS DATA:

$$\Omega(\gamma) \in \mathbb{Z}$$

$$\mu(\gamma_{ij}) \in \mathbb{Z}$$

$$\omega(\gamma, \gamma_a) \in \mathbb{Z}$$

$$\omega(\gamma, \gamma_a + \gamma') = \omega(\gamma, \gamma_a) + \Omega(\gamma) \langle \gamma, \gamma' \rangle$$

PIECEWISE CONSTANT IN STABILITY DATA Z

4. TWISTING FUNCTION

$\sigma(a,b) \in \mathbb{Z}_2$ WHEN $a+b$ DEFINED

$$\sigma(a,b)\,\sigma(a+b,c) = \sigma(a,b+c)\,\sigma(b,c)$$

3 DEFINITIONS

① DEF: A BPS RAY IS A RAY IN CPLX PLANE:

- $\mathbb{R}_- \cdot Z(\gamma)$ IF $\omega(\gamma, \cdot) \neq 0$
- $\mathbb{R}_- \, Z(\gamma_{ij})$ IF $\mu(\gamma_{ij}) \neq 0$

② DEFINE THE (TWISTED) GROUPOID ALGEBRA $\mathbb{C}[\mathbb{V}]$:

$$X_a X_b = \begin{cases} \sigma(a,b) \, X_{a+b} & \text{IF } a+b \text{ COMPOSABLE} \\ 0 & \text{ELSE} \end{cases}$$

3. DEFINE TWO AUTOMORPHISMS OF $\mathbb{C}[\mathbb{V}]$:

- CV-LIKE: $S^{\mu}_{\gamma_{ij}}$

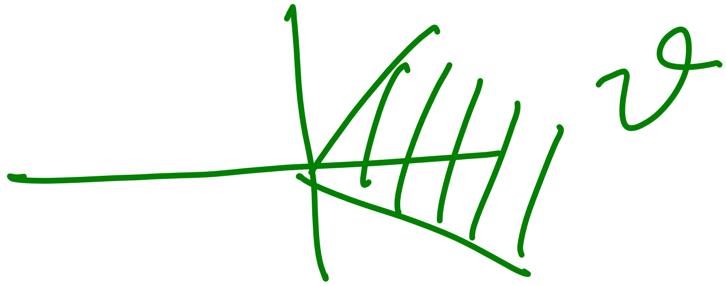
$$X_a \to \left(1-\mu(\gamma_{ij})X_{\gamma_{ij}}\right) \cdot X_a \cdot \left(1+\mu(\gamma_{ij})X_{\gamma_{ij}}\right)$$

- KS-LIKE K^{ω}_{γ} :

$$X_a \longrightarrow \left(1-X_\gamma\right)^{\omega(\gamma,a)} X_a$$

2D/4D WALL-CROSSING FORMULA

FOR CONVEX SECTOR



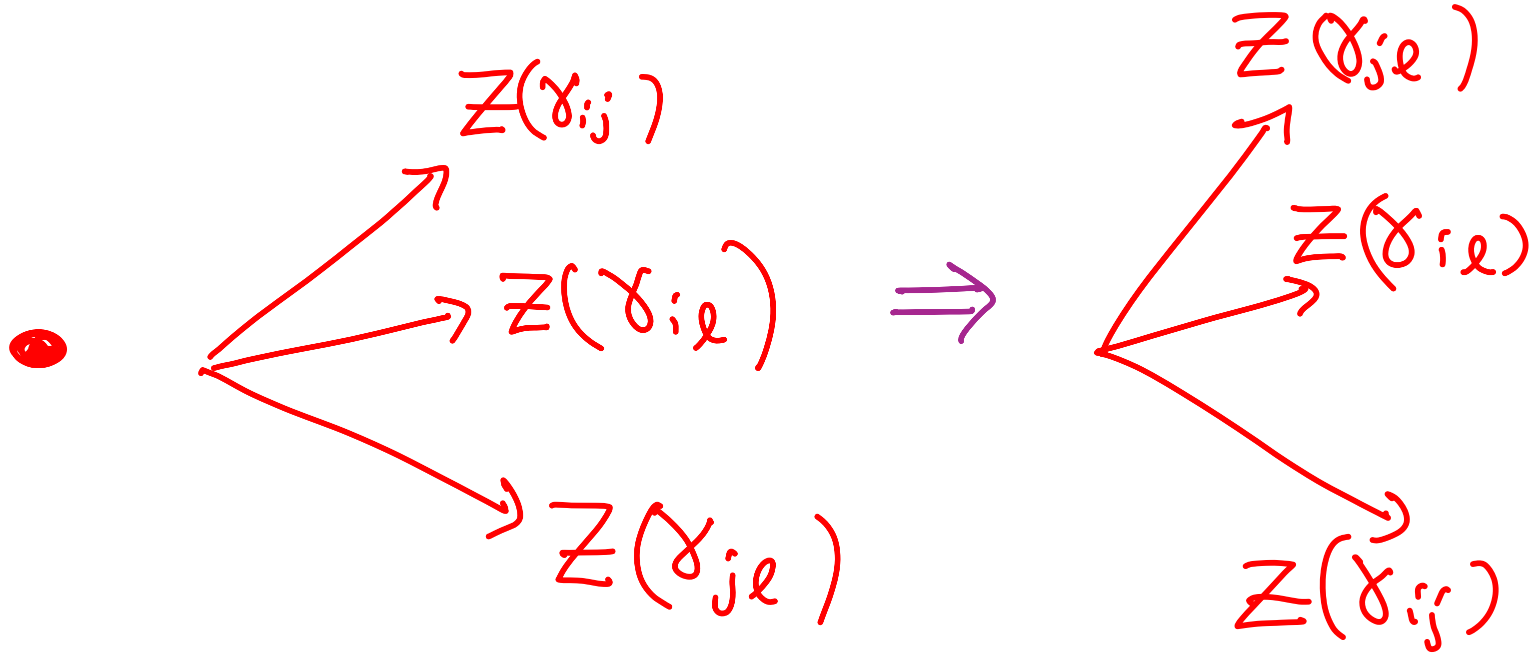
$$A(\vartheta) = :\prod_{Z(\gamma_{ij})\in\vartheta} S_{\gamma_{ij}}^{\mu} \prod_{Z(\gamma)\in\vartheta} K_{\gamma}^{\omega}:$$

ORDERED BY PHASE OF Z

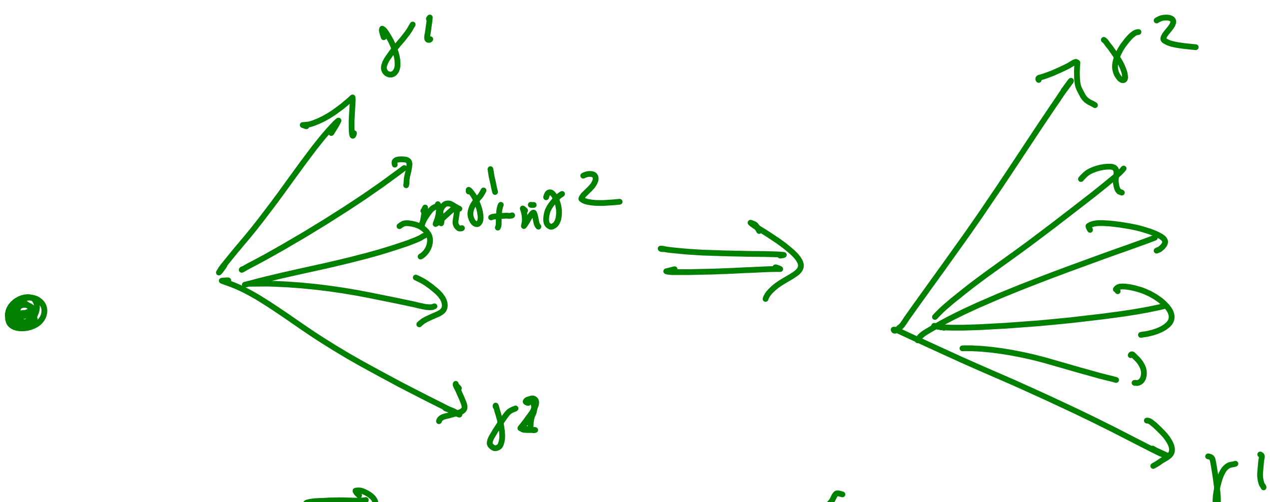
<u>W.C.F</u>: $A(\vartheta)$ IS CONSTANT SO LONG AS NO BPS LINE ENTERS/LEAVES THE SECTOR ϑ.

CV + KS WCF ARE SPECIAL CASES

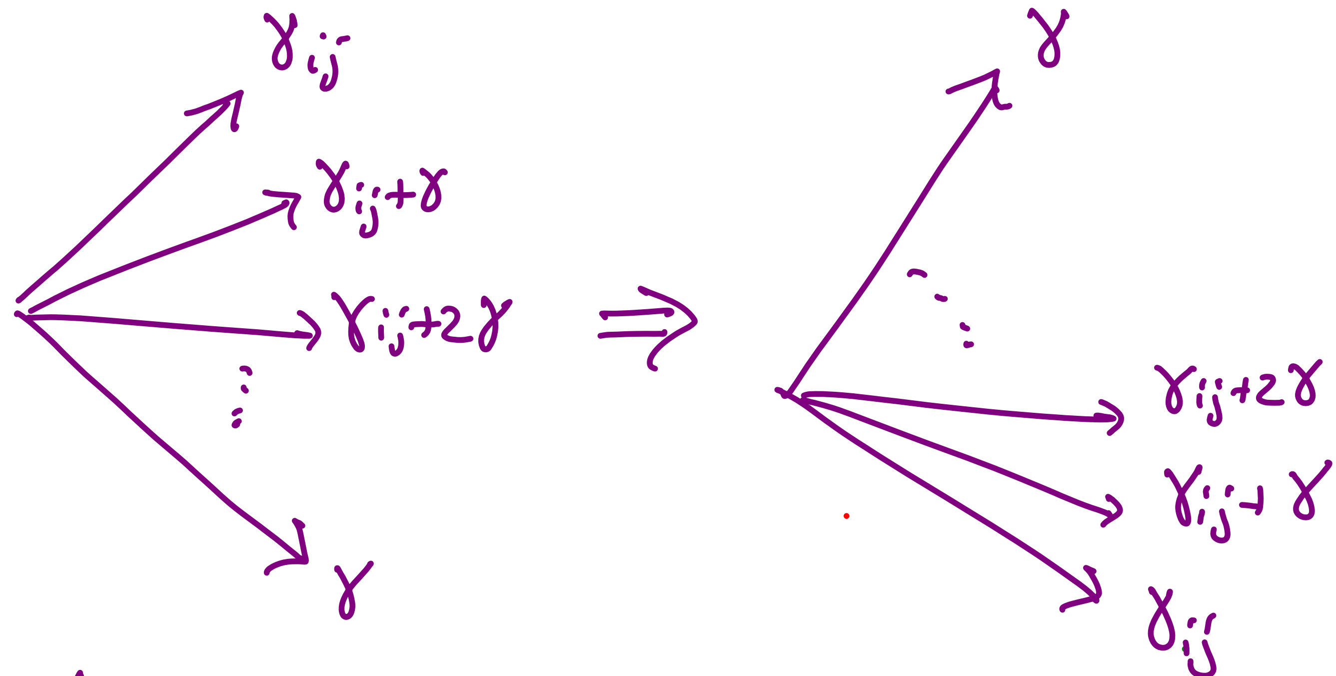
TYPES OF WALL-CROSSING



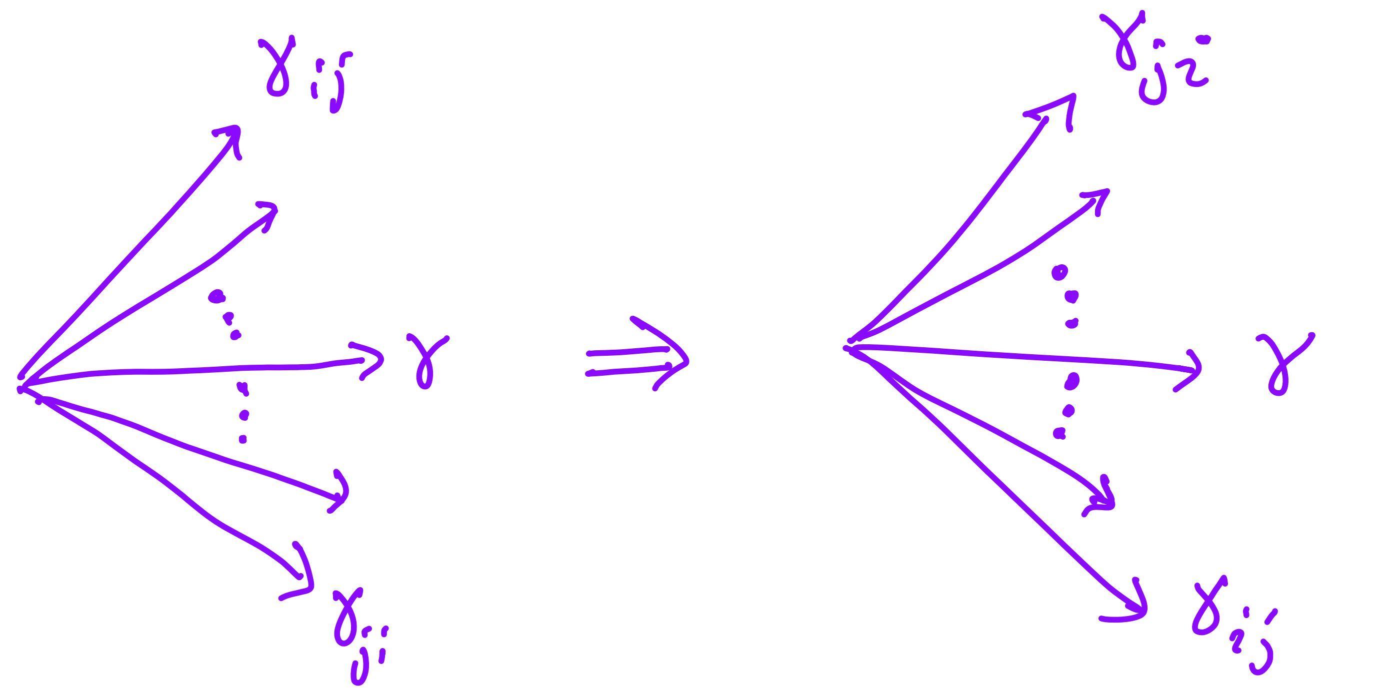
$$S^{\mu}_{\gamma_{ij}}\, S^{\mu}_{\gamma_{il}}\, S^{\mu}_{\gamma_{jl}} = S^{\mu'}_{\gamma_{jl}}\, S^{\mu'}_{\gamma_{il}}\, S^{\mu'}_{\gamma_{ij}}$$



$$\overrightarrow{\prod} K^{\omega} = \overleftarrow{\prod} K^{\omega'}$$

• 

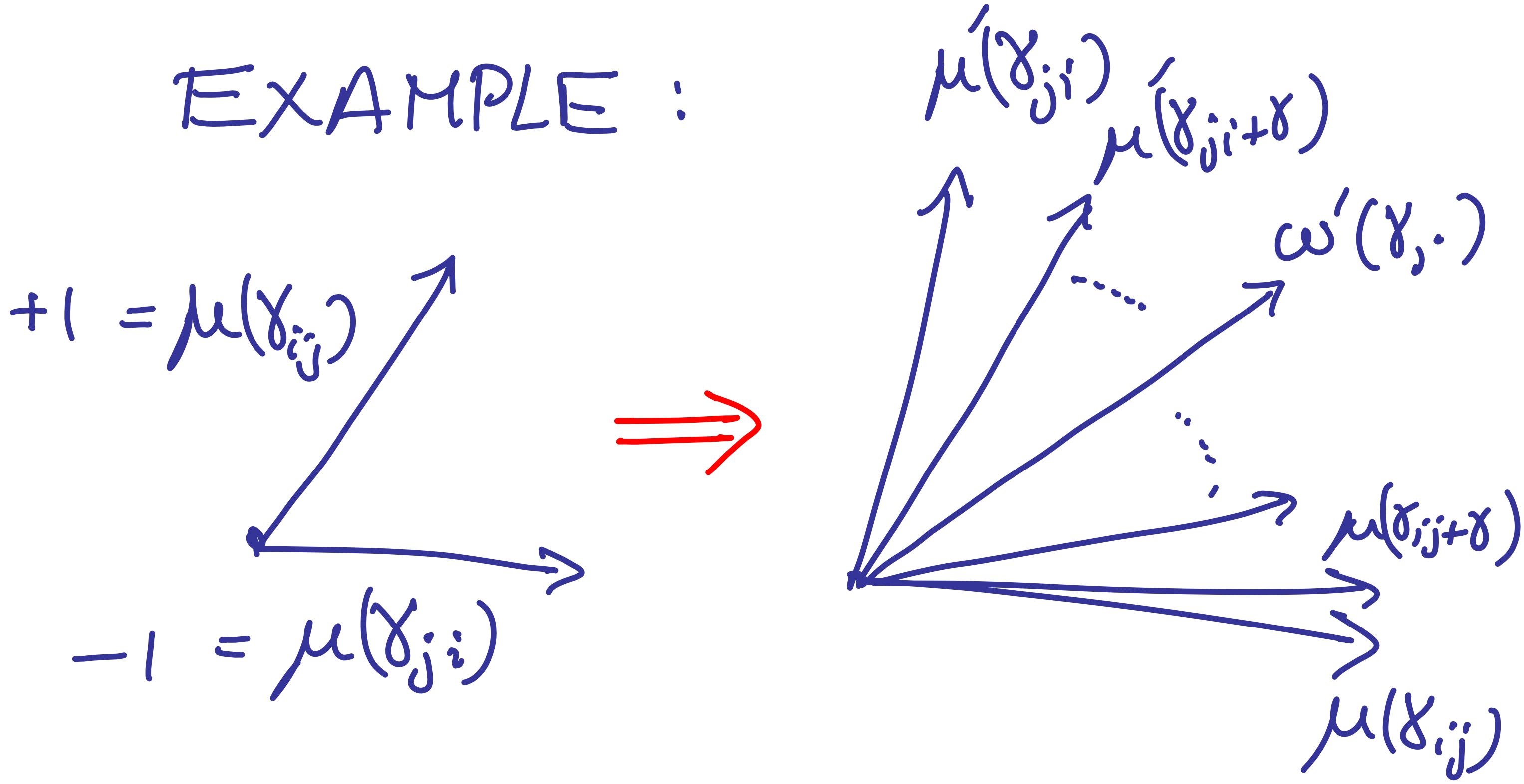
$$K_\gamma^{\omega} \prod S^{\mu}_{\gamma_{ij}+n\gamma} = \prod S^{\mu'}_{\gamma_{ij}+n\gamma} K_\gamma^{\omega'}$$

• 

$$\prod S^{\mu}_{\gamma_{ij}+n\gamma} K_\gamma^{\omega} \prod S^{\mu}_{\gamma_{ji}+n\gamma} = \prod S^{\mu'}_{\gamma_{ji}+n\gamma} K_\gamma^{\omega'} \prod S^{\mu'}_{\gamma_{ij}+n\gamma}$$

THM: MIXED WCF CAN BE SOLVED EXPLICITLY

EXAMPLE :

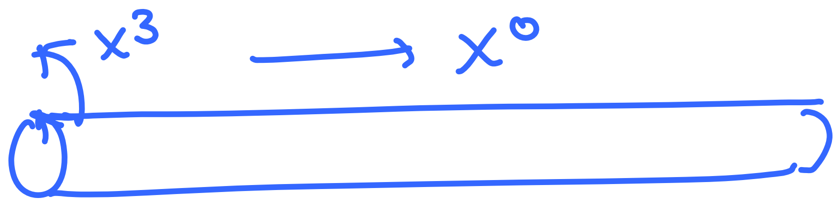


$$S^{\mu}_{\gamma_{ij}} S^{\mu}_{\gamma_{ji}} = \prod^{\nearrow} S^{\mu'}_{\gamma_{ji}+n\gamma} \, K^{\omega'}_{\gamma} \prod^{\searrow} S^{\mu'}_{\gamma_{ij}+n\gamma}$$

ENCODES $\mathbb{CP}^1$ COUPLED TO TWISTED MASS PARAMETER.

(CECOTTI; DOREY)

(H.) 3D GEOMETRY



SURFACE DEFECT IN 4D THEORY

$$\Downarrow$$

LINE DEFECT IN 3D THEORY

$$\exp i \int t_I \left(F_{03}^{I} - i F_{12}^{I}\right) + c.c. \quad dx^0 dx^3$$

$$\Downarrow$$

$$\exp \; i \int dx^0 \left(\eta_I d\theta_{el}^{I} - \alpha^I d\theta_{mg\,I} \right)$$

$$d\theta_{el}^{I} = \oint_{S_R'} F^I \qquad d\theta_{mg,I} = \oint_{S_R'} G_I$$

$\eta_I \, d\theta_{el}^I - \alpha^I d\theta_{mg,I}$ IS A

LOCALLY DEFINED 1-FORM ON $\mathcal{M}$

CROSSING PATCHES WE FIND

A CONNECTION A^{sf} ON

$$(\mathcal{L}_S)_i \longrightarrow \mathcal{M}$$

WHEN $\mathcal{M}$ HAS THE

SEMIFLAT HK METRIC

$$\boxed{A_i^{sf} \text{ IS HYPER-HOLOMORPHIC}}$$

(F_i^{sf} IS TYPE (1,1) IN ALL

COMPLEX STRUCTURES)

QUANTUM CORRECTIONS

$$\begin{tikzcd} \mathcal{L}_{\$} \arrow[r] & \mathcal{M}_{\$} \arrow[r, "\pi"] \arrow[d] & \mathcal{M} \arrow[d] \\ & \mathcal{B}_{s} \arrow[r] & \mathcal{B} \end{tikzcd}$$

$$V_{\$} = \pi_*(\mathcal{L}_{\$}) = \bigoplus_{i \in V} (\mathcal{L}_{\$})_i$$

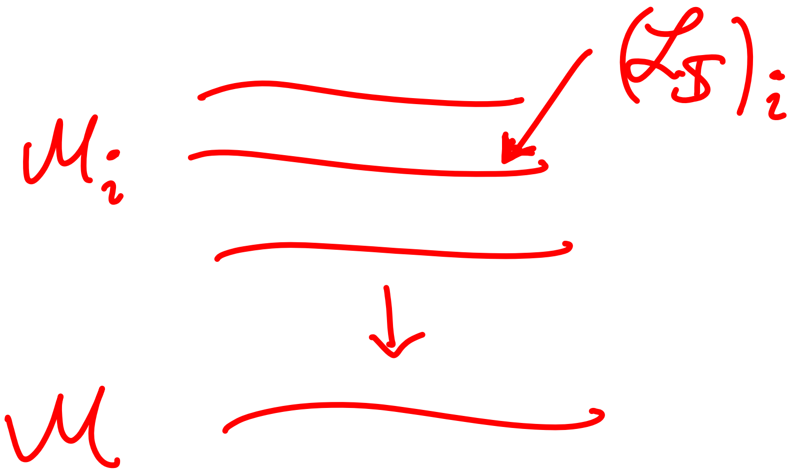
HAS A CONNECTION Θ

3D LINE DEFECT $\quad \mathrm{Tr}\, P\exp \int dx^0\, \varphi^* \Theta$

SUSY $\Rightarrow$ Θ IS HYPERHOLOMORPHIC
(THOMPSON)

CONSTRUCT Θ VIA ITS HOLOMORPHIC SECTIONS

$$\mathcal{Y}_{\gamma_i} \in H^0(V_{\$})$$



Split: $\mathcal{Y}_{\gamma_i} = g_i \, y_{\gamma_i}$

$$y_{\gamma_i} \in H^0\left((\mathcal{L}_S)_i, \mathcal{U}\right)$$

$$g_i \in \mathrm{Hom}\left((\mathcal{L}_S)_i, \bigoplus_j (\mathcal{L}_S)_j\right)$$

$$y_{\gamma_i} = \mathcal{Y}^{sf}_{\gamma_i} \exp\left[-\sum_{\gamma} \omega(\gamma,\gamma_i) K_\gamma * \log(1-X_\gamma)\right]$$

$$\boxed{g_i = g_i^0 - \sum_{\substack{j\neq i \\ \gamma_{ji}}} \mu(\gamma_{ji}) K_{\gamma_{ji}} * \left(g_j \, y_{\gamma_{ji}}\right)}$$

(I.) HITCHIN SYSTEMS

$z \in C \implies \mathbb{S}_z$ SURFACE DEFECT

$V(\mathbb{S}_z) =$ UNIVERSAL BUNDLE $\to \mathcal{M}$

VACUA = PREIMAGES $\Sigma \to C$ (GAIOTTO)

$x_i \to z$

$\Gamma_{ij} = \{\text{Paths } \gamma_{ij} : x_i \text{ TO } x_j\} / \partial(\text{2-CHAIN})$

$$\omega(\gamma, \gamma_{ij}) = \Omega(\gamma; u) \langle \gamma, \gamma_{ij} \rangle$$

γ : WKB Rep.

$\mu(\gamma_{ij}) =$ SIGNED SUM OF WKB CURVES x_i TO x_j THROUGH RAM. POINT OF TYPE (ij)

IX. SOME OPEN PROBLEMS

1. GENERALIZE A_1-THEORIES TO HIGHER RANK: QUALITATIVELY NEW PHENOMENA

2. QUANTIZATION OF $\mathcal{M}$: RELATION TO NEKRASOV-SHATASHVILI & NEKRASOV-WITTEN?

3. APPLICATIONS TO SUGRA?

4. $\mathcal{N}=4$ SCATTERING?

5. NEW MODULAR FUNCTORS.

6. $\overline{\underline{\Omega}}$, PSC, & TOPOLOGY OF MONOPOLE MODULI SPACES.

7. STRONG POSITIVITY & WORK OF LUSZTIG, FOMIN-ZELEVINSKY, ...

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