Quantum monodromy of $4d$ $\mathcal{N} = 2$ theories, TBA and $2d$ RCFT

S.C., Andrew Neitzke, and Cumrun Vafa

arXiv: 1006.3435.

Introduction

from the KS WCF
to the SCFT invariants

KS quantum WCF

Kontsevich Soibelman 0811.23435 ★ Gaiotto, Moore, Neitzke 0807.4723, 0907.3987, 1006.0146 ★ SC Vafa 0910.2615 ★ Dimofte Gukov Soibelman 0912.1346 ★ Andriyash, Denef, Jafferis, Moore 1008.0030

- Γ **charge lattice**
- $\langle\gamma,\gamma'\rangle = -\langle\gamma',\gamma\rangle \in \mathbb{Z}, \quad \gamma,\gamma' \in \Gamma$
- $X_\gamma X_{\gamma'} = q^{\langle\gamma,\gamma'\rangle} X_{\gamma'} X_\gamma$ **quantum torus algebra** $\mathbb{T}_\Gamma \quad (q \in \mathbb{C}^\times)$
- $\Omega(\gamma, s_\gamma) = \begin{cases} \text{signed 'number' of BPS states with charge } \gamma, \\ \text{spin } s_\gamma, \text{ and BPS phase } \exp(i\theta_\gamma) \equiv Z_\gamma/|Z_\gamma| \end{cases}$

Quantum monodromy $M(q)$

$$M(q) = T \prod_{\theta_\gamma} \Psi(q^{s_\gamma} X_\gamma; q)^{\Omega(\gamma, s_\gamma)} \in \mathbb{T}_\Gamma^\times$$

where
- $\Psi(X;q) = \prod_{n=0}^{\infty} (1 - q^{n+1/2} X)$ *'quantum dilogarithm'*
- $T \equiv$ **order product in increasing BPS phase** $\theta_{\mathrm{BPS}} \equiv$ **time–order**

WCF $\Leftrightarrow$ the conjugacy class of $M(q)$ in $\mathbb{T}_\Gamma^\times$ is a chamber–mutation invariant

Our problem: study the WC invariants

$$\mathrm{Tr}\, M(q)^k \qquad k \in \mathbb{Z}$$

certain fractional values of k also are good invariants
$\Rightarrow$ fractional monodromy $k = n/h$ (h a 'Coxeter number')

Physical meaning

$\mathcal{N} = 2$ SCFT $\longrightarrow$ deform in the IR by masses/Coulomb branch vev's $\longrightarrow$ non–trivial BPS spectrum depending on the particular deformation

$\mathrm{Tr}\, M(q)^k$ an invariant which does not depend on the chosen deformation $\longrightarrow$ a property of the UV SCFT

Example: $d = 2$

Similar WCF with $M \in SL(n)$ (n= number of SUSY vacua)

$$M = T \prod_{\theta_\gamma} (\mathbf{1} + E_\gamma)^{\Omega(\gamma)}$$

- γ **a root of** $\mathfrak{sl}(n)$
- E_γ **the matrix representing** γ **in the fundamental rep.**
- $\Omega(\gamma)$ $\left\{ \begin{array}{l} \textbf{the signed `number' of BPS particles in the } \gamma \textbf{ sector} \\ \textbf{of the Hilbert space with BPS phase } \exp(i\theta_\gamma) = Z_\gamma/|Z_\gamma| \end{array} \right.$

$$\mathrm{Tr}\, M^k = \sum_{RR \text{ vacua}} e^{2\pi i k\, R} \Big|_{\textbf{UV SCFT}} \qquad R: \; SCFT \;\, U(1) \text{ charge}$$

The LHS is computed in the mass–deformed theory from the BPS spectrum; the RHS is a superconformal invariant (identifying the UV SCFT: c, h,...)

Inverse problem:

Given the SCFT, determine the possible BPS mass spectra of the deformed theory (up to WCF equivalence)

Classification problem:

Use the relations between BPS spectra and the SCFT data to set constraints on both, and then determine the set of consistent $\mathcal{N} = 2$ theories

In $d = 2$ both problems were analyzed (to some extend)
see: SC & Vafa `arXiv:hep-th/9204102`.

$\Rightarrow$ do the same in the $d = 4$ case

A simple set–up

(just to motivate the construction,
the arguments and results are fully general, and apply
also when the following discussion does not)

- M–theory on

$$\mathbb{R}^4 \times \mathbb{C}^2_{x,y} \times \left(\mathbb{C}_z \times \mathbb{R}_p\right)$$

with a $M5$ brane wrapped on

$$\mathbb{R}^4 \times \Sigma \times \{z=0,\ p=0\}$$

$$\Sigma \equiv \{f(x,y)=0\} \subset \mathbb{C}^2_{x,y}$$

On $\mathbb{R}^4$ a $\mathcal{N}=2$ theory with SW curve Σ and differential $\lambda = y\,dx$

The theory is an $\mathcal{N}=2$ SCFT $\Rightarrow$ the SW curve Σ is singular

$$\textbf{E.g.} \quad f(x,y) = y^m - x^n = 0.$$

- **Duality:** Compactify the theory by replacing $\mathbb{R}^4 \to \mathbb{R}^2 \times S^1 \times S^1$

g: isometry of $\mathbb{C}^2_{x,y}$ preserving Σ (symmetry of $\mathcal{N}=2$ 4d theory)

$$\mathcal{M}:\ \mathbb{R}^2 \times \mathbf{S^1} \times \left(S^1 \times_g \mathbb{C}^2_{x,y}\right) \times \left(\mathbb{C}_z \times \mathbb{R}_p\right)$$

$$M5:\ \mathbb{R}^2 \times S^1 \times \left(S^1 \times_g \Sigma\right) \times \{z=0,\ p=0\}$$

$$\Longrightarrow \quad \mathbf{IIA}:\ \mathbb{R}^2 \times \left(S^1 \times_g \mathbb{C}^2_{x,y}\right) \times \left(\mathbb{C}_z \times \mathbb{R}_p\right) = \left(\mathbb{R}^2 \times \mathbb{C}_z\right) \times K$$

$$D4:\ \mathbb{R}^2 \times \left(S^1 \times_g \Sigma\right) \times \{z=0,\ p=0\} = \left(\mathbb{R}^2 \times \{z=0\}\right) \times L$$

$K = (S^1 \times_g \mathbb{C}^2_{x,y}) \times \mathbb{R}_p$ (**CY**) $\qquad L = (S^1 \times_g \Sigma) \subset K$ (**Lagrangian**)

By the duality chain the $\mathcal{N}=2$ g–twisted partition function gets mapped into the topological A model on K with a brane on L

$$\longrightarrow \quad Z^{\rm open}_{\rm top}(K,L) = {\rm Tr}_{\rm CS}\big[M\big]$$

$$M = T\prod_\alpha \mathcal{O}_\alpha \qquad \mathcal{O}_\alpha \text{ instanton corrections}$$

$$g^r = 1 \quad \Leftrightarrow \quad M^r = 1.$$

E.g. $f(x,y) = y^m - x^n$. Choose g to correspond to $\exp(2\pi i R)$

$$g: \ (x,y) \to (\omega^m x, \omega^n y) \qquad \text{with } \omega^{m+n} = 1$$

which leaves λ invariant

$$r = (m+n)/\gcd(m,n) \qquad g^r = 1$$

Canonical singularities:

singularity	$f(x,y)$	$r =$ order g
A_{n-1}	$y^2 - x^n$	$\begin{cases} n & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$
D_n	$x^{n-1} + xy^2$	$\begin{cases} n & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$
E_6	$x^3 + y^4$	7
E_7	$x^3 + xy^3$	5
E_8	$x^3 + y^5$	8

Next deform the SW curve Σ by lower order monomials to make it non–singular $\rightarrow$ $4d$ $\mathcal{N}=2$ no–longer SCFT, non–trivial BPS spectrum:

- $\Omega(\gamma)$= signed 'number' BPS state of given charge $\gamma \in \Gamma$

$\mathrm{Tr}\, M^k$ is a SUSY index <u>invariant</u> under deformations
It has the general form

$\mathrm{Tr}_{\mathrm{CS}}\Big[T \prod \mathcal{O}_\alpha\Big]$

$\mathcal{O}_\alpha$ operator insertion corresponding
to instantons of the open topological theory
$\equiv$ holomorphic curves $\mathcal{C}$ with $\partial\mathcal{C} \subset L$
$\equiv$ **BPS states of the $4d$ $\mathcal{N}=2$ theory**

Topological strings $\Rightarrow$

$$\mathcal{O}_\alpha = \Psi\Big(q^{s_{\gamma_\alpha}}\, X_{\gamma_\alpha},\; q\Big)^{\Omega(\gamma_\alpha,\, s_{\gamma_\alpha})}$$

and one recovers the quantum WCF.

In this framework we can address our three problems:

- **DIRECT**: For A_{n-1} we know the BPS spectrum (in some 'canonical' chamber) $\Rightarrow$ compute the SCFT invariant data;
- **INVERSE**: For $A-D-E$ models we know the UV order r. Then we have an equation

$$\left(T\prod_{\gamma}\Psi(X_{\gamma};\,q)^{\Omega(\gamma)}\right)^{r} = \text{Identity on } \mathbb{T}_{\Gamma}^{\times} \text{ for all } q\in\mathbb{C}^{\times}$$

  which we can solve for $\Omega(\gamma)$ (and BPS phase–order). Thus we get the BPS spectrum (up to WCF mutation equivalence) of $A-D-E$ theories from the UV datum r;
- **CLASSIFICATION**: Find all *integers* $\{r,\Omega(\gamma)_{\gamma\in\Gamma}\}$ such that the above identity holds for all $q\in\mathbb{C}^{\times}$. In particular, it must hold as $q^{1/2}\to\pm1$ (classical limits) $\Rightarrow$ **Bloch group in number theory, Nahm conjectures in RCFT, etc.**

Before going to that, let us consider **more general** theories

Previous ADE theories obtained by compactifying IIB on the local CY

$$f(x,y)+u^2=v^2=0 \qquad (*)$$

we can consider more general hypersufaces $W(x,y,u,v)=0$
W quasi–homogeneous $W(\lambda^{q_i}x_i)=\lambda\, W(x_i)$ $(\forall\lambda\in\mathbb{C})$: It corresponds to a singularity at finite distance in CY moduli space iff
$\hat{c}:=4-2\sum_{i=1}^{4}q_i<2$ *i.e.* $\sum_{i=1}^{4}q_i>1$
$(*)$ automatically satisfies the condition. Other class of solutions

$$W(x,y,u,v)=W_G(x,y)+W_{G'}(u,v)$$

where $G=ADE$ and $W_G(x,y)$ is the canonical singularity associated to the given simply–laced Lie algebra. (G,G') models, the previous ADE models being (G,A_1). For the (A_{n-1},A_{m-1}) model, $x^n+y^m+u^2+v^2$

$$r=(n+m)/\gcd(n,m)$$

$$\text{general case} \quad r=\begin{cases}\frac{1}{4}\frac{h(G)+h(G')}{\gcd(h(G)/2,h(G')/2)} & G,G'=A_1,D_{2n},E_7,E_8\\ \frac{h(G)+h(G')}{\gcd(h(G),h(G'))} & \text{otherwise}\end{cases}$$

FRACTIONAL MONODROMY assume we may deform $\Sigma_{\rm sing}$ to a smooth curve (or, more generally, a smooth local 3–CY) while preserving some discrete $\mathbb{Z}_k$ R–symmetry (this assumption selects a class of deformations and hence of compatible BPS chambers) which acts on the central charge as $Z \to e^{2\pi i/k} Z$ and on $\mathbb{T}_\Gamma$ by some operator R
$X_\gamma \to R\, X_\gamma\, R^{-1}$ with $R^k = 1$

$$E.g. \quad y^k + x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

$$y \to e^{2\pi i/k} y, \quad x \to x \qquad \lambda \equiv y\, dx \to e^{2\pi i/k}\, \lambda$$

$$\text{set} \quad KS(\theta', \theta) = T \prod_{\theta_{\rm BPS}=\theta}^{\theta'} \Psi(X_\gamma; q)^{\Omega(\gamma)}$$

$$\mathbb{Z}_k \text{ symm.} \quad KS(\theta' + 2\pi/k,\, \theta + 2\pi/k) = R\, KS(\theta',\, \theta)\, R^{-1}$$

$$\text{then} \quad M(q) = \left(R^{-1}\, KS(2\pi/k, 0)\right)^k$$

$$Y(q) = R^{-1}\, KS(2\pi/k, 0) \qquad \frac{1}{k}\text{–monodromy}$$

Concretely:

- Our 'Diophantine' problems $M^r = 1$ may be put in a 'canonical' form using the machinery of Cluster algebras;
- The 'canonical' formulation implies a beautiful and unexpected connection with the corresponding $d = 2$ problems.

For simplicity, I will limit myself to the **INVERSE** problem for the (G, G') $(G, G' = ADE)$. This means, given r, find Ω's and BPS–phase order (in some chamber). In practice

- find a particular solution
- argue that it is the unique one up to KS WCF equivalence.

Cluster algebras a useful tool to construct (particular?) solutions

★ QUIVERS

To $\mathbb{T}_\Gamma$ associate a 2–acyclic quiver Q_Γ:

- to each generator of Γ, γ_i, $(i = 1, 2, \ldots, \ell)$ associate a **node**
- if $\langle\gamma_i, \gamma_j\rangle > 0$ we draw $\langle\gamma_i, \gamma_j\rangle$ arrows from node i to node j
- if $\langle\gamma_i, \gamma_j\rangle < 0$ we draw $|\langle\gamma_i, \gamma_j\rangle|$ arrows from node j to node i

$$B_{ij} = -B_{ji} \equiv \langle\gamma_i, \gamma_j\rangle \qquad \text{exchange matrix of the quiver } Q_\Gamma$$

• ADE models $(G, A_1) \to$
Dynkin Quiver $\equiv$ Dynkin diagram of G with only sources & sinks

• General (G, G') models $\to$
Quiver $\equiv$ **square product** of the G, G' Dynkin quivers

$$G \,\square\, G' = (G' \,\square\, G)^\vee \qquad B^\vee = -B$$

To each node of Q_Γ associate the corresponding element of $\mathbb{T}_\Gamma$

$$X_i \equiv X_{\gamma_i}$$

★ QUANTUM MUTATIONS

A **quantum mutation** in $\mathbb{T}_\Gamma$ is an (ordered) sequence of **elementary** mutations.

Elementary mutation $\mathcal{Q}_k$ at the k–th node composition of **two** transformations:

1. a change of basis in $\Gamma \equiv$ a QUIVER MUTATION (Seiberg duality, 2d WCF, PL,....)

$$X_i \to X_i' = q^{-\langle\gamma_i,\gamma_k\rangle\,[\langle\gamma_i,\gamma_k\rangle]_+/2}\,X_i\,X_k^{[\langle\gamma_i,\gamma_k\rangle]_+} \qquad i \neq k$$
$$X_k \to X_k' = X_k^{-1} \qquad \text{where } [a]_+ \equiv \max\{a,0\}$$

2. the adjoint action of the the quantum dilogarithm of $X_k \equiv X_{\gamma_k}$

$$X_\gamma \mapsto \Psi(X_k;q)^{-1}\,X_\gamma\,\Psi(X_k;q)$$

As $q^{1/2} \to \pm 1$, **classical cluster mutations**

A **cluster quantum mutation**

$$\mathbf{M} = \overrightarrow{\prod} \mathcal{Q}_k$$

is a solution to our 'Diophantine' problem *(a putative quantum monodromy of a $\mathcal{N} = 2$) model* if

1. $\mathbf{M}$ may be written has a product of adjoint actions of $\Psi(X_\gamma; q)^{\pm 1}$ alone (namely if the corresponding product of base changes/Seiberg dualities is the identity on $\mathbb{T}_\Gamma$);
2. $\mathbf{M}^r = \mathbf{1}$ on $\mathbb{T}_\Gamma$

Assume there is a cluster quantum mutation $\mathbf{Y}$ such that

$$\mathbf{M} = \mathbf{Y}^h, \qquad h \in \mathbb{N}$$

Then $\mathbf{Y}$ is naturally identified with a $1/h$–fractional monodromy. $\mathbf{Y}$ has general structure:

$$\begin{aligned} \mathbf{Y} &= (\text{a base change in } \mathbb{T}_\Gamma) \times (\text{ordered product of } \Psi(X_\gamma; q)^{\pm 1}) \\ &= \qquad R^{-1} \qquad \times \qquad KS(2\pi/h, 0) \end{aligned}$$

Relation to TBA

Lemma. *If Q_Γ is simply–laced (i.e. $|B_{ij}| \leq 1$) $\mathbf{M}^r = \mathbf{1}$ iff the corresponding* <u>*classical*</u> *cluster mutation $\mathbf{M}_{\rm class} \equiv \mathbf{M}\big|_{q\to 1}$, has order r.*

Let us consider the classical cluster algebra associated to the simply–laced quiver $G \,\square\, G'$ (G, G' a pair of ADE Dynkin diagrams)

Theorem. *The classical cluster algebra of $G \,\square\, G'$ posseses two* INVOLUTIVE *mutations E and O such that setting*

$$Y_{k,a}(s) = \begin{cases} O\cdot Y_{k,a}(s-1) & s \text{ odd} \\ E\cdot Y_{k,a}(s-1) & s \text{ even}\end{cases} \qquad \begin{matrix} k = 1,\cdots,\mathrm{rank}\, G \\ a = 1,\cdots,\mathrm{rank}\, G' \end{matrix}$$

$Y_{k,a}(s)$ is the solution to Zamolodchikov (G,G') TBA Y–system

$$Y_{k,a}(s+1)\; Y_{k,a}(s-1) = \frac{\prod_{j\neq k}\big(1+Y_{j,a}(s)\big)^{-C_{kj}}}{\prod_{b\neq a}\big(1+Y_{k,b}(s)^{-1}\big)^{-C'_{kj}}}$$

$C_{kj},\ C'_{a,b} =$ Cartan of $G,\ G'$. *The mutation $\mathbf{Y}_{\rm class} = E\, O$ has order*

$$r = \begin{cases} \frac{1}{2}\Big(h(G)+h(G')\Big) & G, G' = A_1, D_{2n}, E_7, E_8 \\ h(G)+h(G') & \text{otherwise}\end{cases}$$

Y quantum cluster mutation w/ classical limit $\mathbf{Y}_{\text{class}}$
By lemma + theorem, the quantum cluster mutation

$$\mathbf{M} = \mathbf{Y}^{h(G')}$$

is a solution to Diophantine problem $\Leftrightarrow$ $\mathbf{M} = \mathrm{Ad} \prod \Psi(X_\gamma; q)$
$\Leftrightarrow$ associated quiver mutation = **identity**. **TRUE**

- **M** quantum monodromy of (G, G') model
- **Y** $\frac{1}{h(G')}$–fractional monodromy:
  local 3–CY geometry $\Rightarrow$ $\exists$ $\mathbb{Z}_{h(G')}$–symmetric chambers

Solution to **INVERSE PROBLEM**
Agreement with direct computation of BPS spectrum for (A_m, A_1)
In some $\mathbb{Z}_{h(G')}$*–symmetric chamber for each node of the* G *Dynkin diagram there are BPS states in one–to–one correspondence with the roots of the Lie algebra* G'

$$\#\,\mathrm{BPS} = \mathrm{rank}\,G \cdot \mathrm{rank}\,G' \cdot h(G') = \mathrm{rank}\,G \cdot \#\{\text{roots of } G'\}$$

'LEVEL–RANK' DUALITY

The local 3–CY geometry

$$W_G(x,y) + W_{G'}(u,v) = 0$$

symmetric under $G \leftrightarrow G'$. $\mathbf{M}$ *NOT* manifestly invariant

Quivers $G \,\square\, G'$ and $G' \,\square\, G$ correspond to *different* chambers w/ resp. $\mathbb{Z}_{h(G')}$ & $\mathbb{Z}_{h(G)}$ symmetry

E.G. (A_2, A_1) 2 BPS states, (A_1, A_2) 3 BPS states.

Monodromies should be equal **up to conjugacy**. TRUE ?

$$\mathbf{Y}_{\rm class} = E\, O, \qquad \mathbf{Y}^\vee_{\rm class} = O\, E = \mathbf{Y}^{-1}_{\rm class}$$

$$\Rightarrow \quad \mathbf{M}_{G' \square G} = (\mathbf{Y}^\vee)^{h(G)} = \mathbf{Y}^{-h(G)} = \mathbf{Y}^{h(G)+h(G')-h(G)} =$$

$$= \mathbf{Y}^{h(G')} = \mathbf{M}_{G \square G'}$$

E.G. $(A_2, A_1) = (A_1, A_2) \Leftrightarrow$ **Pentagonal identity**

$(G, G') = (G', G)$ higher quantum dilogarithm identities

$\xrightarrow{\mathbf{q} \to \mathbf{1}}$ **Bloch group**

$\Rightarrow$ **Level–rank duality of $2d$ RCFT**

TBA & 2*d* (MASSIVE) INTEGRABLE MODELS

Y–systems introduced by Zamolodchikov to solve the TBA equation for 2*d* massive integrable systems w/ elastic S–matrices (originally *ADE* integrable models)

$$Y_{k,a}(s) := e^{-\epsilon(\theta - is\pi/h)_{k,a}}$$

$\epsilon(\theta)_{k,a}$	pseudoenergy of (k,a) particle
θ	rapidity

Y–system together with zeros and boundary condition at $\theta \to \infty$ uniquely determine the solution

TBA integral equation $\longrightarrow$ essentially the same integral equation of Gaiotto, Moore, Neitzke giving the hyperKähler geometry of the 3*d* sigma model obtained by compactifing the 4*d* $\mathcal{N} = 2$ on S^1

RCTF

Our original problem: study the WCF invariants

$$\mathrm{Tr}\, Y^k \equiv \mathrm{Tr}\, M(q)^{k/h}$$

It is a q–series. From

$$\Psi(z;q) = \sum_{n\geq 0} \frac{q^{n^2/2}(-z)^n}{(q)_n} \qquad (q)_n = \prod_{k=1}^{n}(1-q^k)$$

it must have the general structure

$$\sum_{\mathbf{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\mathbf{m}\cdot A\cdot\mathbf{m}+B\cdot\mathbf{m}+C}}{\prod_{i=1}^{\ell}(q)_{m_i}^{a_i}}$$

for certain $\ell \times \ell$ matrix A, vector B, integral vector a_i, and C

Typical of RCFT characters (up to q–hypergeometric identities)

In particular, $k = 1$ for (G, A_1) models

$$\mathrm{Tr}\, Y = \sum_{\mathbf{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\mathbf{m}\cdot A\cdot\mathbf{m}+B\cdot\mathbf{m}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_\ell}} \qquad A = \frac{1}{2}C_G \text{ Cartan of } G$$

(B depends on the particular definition of trace, for the standard one $B = 0$)

More generally

$$\mathrm{Tr}\big[Y\prod_\alpha X_{\gamma_\alpha}\big] = \sum_{\mathbf{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\mathbf{m}\cdot A\cdot\mathbf{m}+B\cdot\mathbf{m}+C}}{(q)_{m_1}(q)_{m_1}\cdots(q)_{m_\ell}}$$

where B and C depend on the insertions and $A = C_G/2$

$\Rightarrow$ CHARACTERS OF THE GENERALIZED PARAFERMIONS i.e. the coset RCFT

$$(\hat{G})_2/U(1)^\ell, \qquad \ell = \mathrm{rank}\, G$$

The massive integrable $2d$ G model ($G = ADE$) has as UV fixed point precisely the parafermion theory $\mathbf{(\hat{G})_2/U(1)^\ell}$!!

Then one expects that

$$\mathrm{Tr}\Big[Y\prod_\alpha X_{\gamma_\alpha}\Big]\Big|_{(G,G')\ \text{model}} = \left[\begin{array}{l}\text{linear combination of characters}\\ \text{of the UV RCFT corresponding to}\\ \text{the integrable massive } (G,G')\text{ model}\end{array}\right.$$

$$= \sum_{\mathbf{m}\in\mathbb{N}^\ell} \frac{q^{\frac{1}{2}\mathbf{m}\cdot A\cdot\mathbf{m}+B\cdot\mathbf{m}+C}}{(q)_{m_1}(q)_{m_1}\cdots(q)_{m_\ell}}$$

$$\ell = \mathrm{rank}\, G \times \mathrm{rank}\, G' \qquad \mathbf{A = C_G \otimes C_{G'}^{-1}}$$

- ✓ explicit computation for $(G; A_1)$
- ✓ limit $q \to 1$
- ✓ recursion relations for different operator insertions
- ✓ symmetries and central charges
- ✓ **relation with Nahm's conjectures in** $2d$

E.g. for (G, A_m) model we get the RCFT coset

$$\hat{\mathbf{G}}_{\mathbf{m+1}}/\mathbf{U(1)}^{\mathbf{rank\,G}}$$

Then

$$(A_n, A_m) \leftrightarrow (A_m, A_n)$$
$$\widehat{SU(n+1)}_{m+1}/U(1)^n \leftrightarrow \widehat{SU(m+1)}_{n+1}/U(1)^m$$

'usual' rank–level duality

$\Rightarrow$ for $n = 2, m = 1$ implies usual pentagonal identity

• general rank–duality additional quantum dilogaritm identities

General case $\mathrm{Tr}\, Y^k$

One gets again RCFT characters (possibly non–unitary theory) possibly times ‘trivial’ factors $1/\eta(q)^s$

E.g. for the (A_2, A_1) model one gets the full family of $(*, 5)$ **minimal models** [$5 =$ order of Y !]

$$(p, p') = (2, 5),\ (3, 5),\ (4, 5)$$

Writing $\mathrm{Tr}\, Y^k$ in different chambers one gets many (new) Rogers–Ramanujan–like identities between q–series/∞ products

- A nice pattern emerges $\Rightarrow$ **work in progress !!**
- Relations **cluster algebra** *vs.* **Verlinde algebra**
- further details in `arXiv: 1006.3435`

Thank You