D-instantons, Quantum mirror symmetry and Integrability

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Montpellier

Plan of the talk

1. Compactified string theories and their symmetries

2. Twistor description of non-perturbative moduli spaces

   S.A., B.Pioline, F.Saueressig, S.Vandoren arXiv:0812.4219

   S.A. arXiv:0902.2761

3. Non-perturbative mirror map and SL(2,$\mathbb{Z}$) symmetry in the twistor space

   S.A., F.Saueressig arXiv:0906.3743

4. Integrability of instanton corrections and Thermodynamic Bethe Ansatz

   S.A., Ph. Roche arXiv:1003.3964

5. Conclusions

Compactifications of string theory

Type II superstring theory compactified on a Calabi-Yau

⬇

effective theory in M_4:

$\mathcal{N}$=2 supergravity in 4d coupled to matter

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The low-energy action is determined by the geometry of the *moduli space* parameterized by scalars of *vector* and *hypermultiplets:*

$$\boxed{\begin{array}{c}\mathcal{M}_{VM}\\ z^a,\ a{=}1,\dots,n_V\end{array}} \times \boxed{\begin{array}{c}\mathcal{M}_{HM}\\ q^\alpha,\ \alpha{=}1,\dots,4n_H\end{array}}$$

- $\mathcal{M}_{VM}$ – *special Kähler*
- determined by *holomorphic prepotential* $F(z)$
- no corrections in string coupling g_s

- $\mathcal{M}_{HM}$ – *quaternion-Kähler*
- can be constructed via its *twistor space*
- receives all types of g_s-corrections

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The goal: to find the non-perturbative geometry of $\mathcal{M}_{HM}$

Symmetries and dualities

Type IIA/X

$$\mathcal{M}_{VM} = \mathcal{K}_K(X)$$

$$\mathcal{M}_{HM} = \mathcal{M}_C(X)$$

Type IIB/Y

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Symplectic invariance **SL(2, $\mathbb{Z}$)**

Mirror symmetry

$$\mathrm{IIA}/X \;=\; \mathrm{IIB}/\widetilde{X}$$

$$Y = \tilde{X}$$

CY *mirror* to X

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Symplectic invariance

c-map

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SL(2, $\mathbb{Z}$)

Mirror symmetry

IIA/X $=$ IIB/$\widetilde{X}$

$\mathcal{K}_K(X) = \mathcal{K}_C(\tilde{X})$

well established

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known only *classically*

$(\alpha' \to 0,\ g_s \to 0)$

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Classical mirror map

Bohm,Gunther,Hermann,Louis ’99

VM moduli $z^a = b^a + \mathrm{i}t^a$

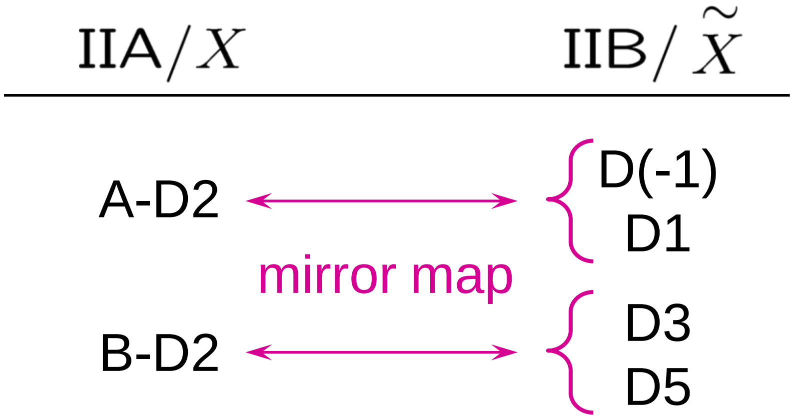
RR fields
$$\begin{aligned} \zeta^0 &= \tau_1 \\ \zeta^a &= -(c^a - \tau_1 b^a) \\ \tilde{\zeta}_a &= c_a + \tfrac{1}{2}\,\kappa_{abc}\, b^b (c^c - \tau_1 b^c) \\ \tilde{\zeta}_0 &= c_0 - \tfrac{1}{6}\,\kappa_{abc}\, b^a b^b (c^c - \tau_1 b^c) \end{aligned}$$

NS axion $\sigma = -2(\psi + \tfrac{1}{2}\tau_1 c_0) + c_a (c^a - \tau_1 b^a) - \tfrac{1}{6}\,\kappa_{abc}\, b^a c^b (c^c - \tau_1 b^c)$

Quantum corrections

Corrections to the classical HM moduli space:

- worldsheet instantons
- one loop g_s-corrections
- D-brane instantons



IIA/X $\mathrm{IIB}/\widetilde{X}$

A-D2 ⟷ D(-1), D1

mirror map

B-D2 ⟷ D3, D5

- NS5-brane instantons

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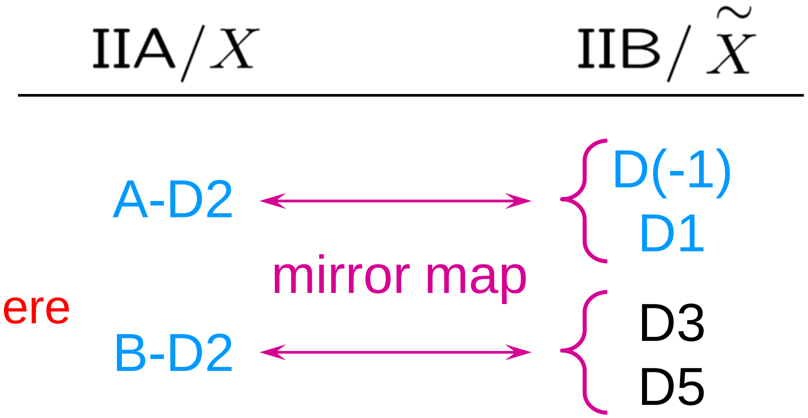

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A-D2 ⟷ { D(-1), D1 } — Robles-Llana,Rocek,Saueressig, Theis,Vandoren ’06, ‘07

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we are here

S.A.,Pioline,Saueressig, Vandoren ’08

- NS5-brane instantons

partial results with D.Persson, B.Pioline
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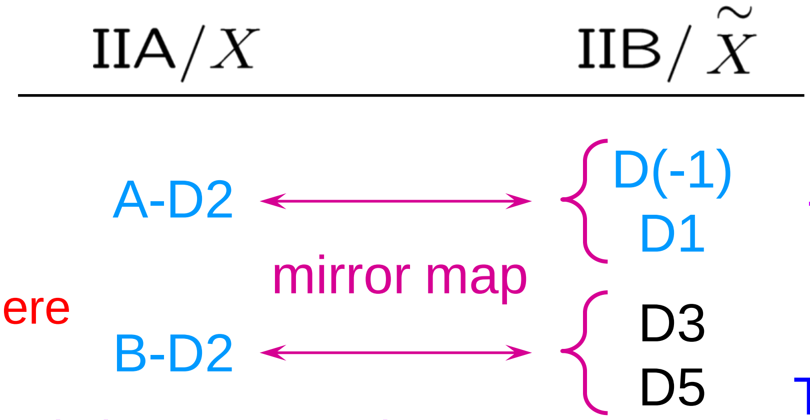

— Robles-Llana,Rocek,Saueressig, Theis,Vandoren ’06, ‘07

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To go further we need an ***instanton corrected mirror map***

- NS5-brane instantons

partial results with D.Persson, B.Pioline
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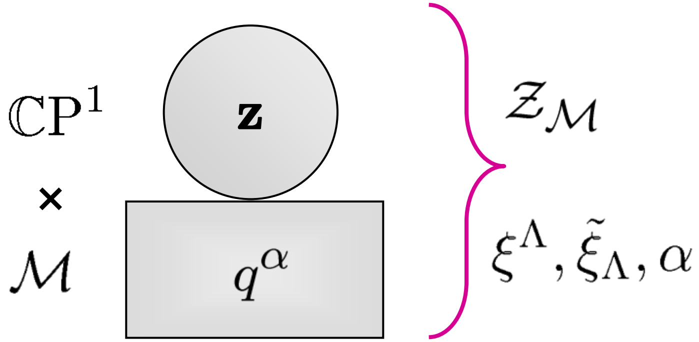
I’ll present corrections due to ***worldsheet, D(-1) and D1 instantons***

Twistor space formulation

How to conveniently describe (parametrize) a quaternion-Kähler manifold?

Twistor space



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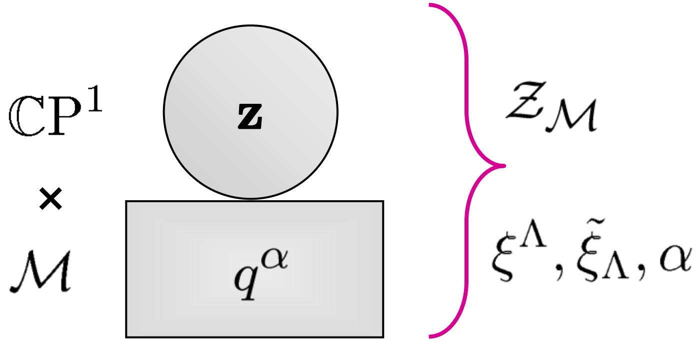

Properties:

- Einstein-Kähler
- has odd complex dimension
- carries *contact structure*

$$\mathcal{X}^{[i]} \equiv \mathrm{d}\alpha^{[i]} + \xi^\Lambda_{[i]}\, \mathrm{d}\tilde\xi^{[i]}_\Lambda$$

complex Darboux coordinates

$$\text{such that} \quad \mathcal{X}^{[i]} \wedge (\mathrm{d}\mathcal{X}^{[i]})^d \neq 0$$

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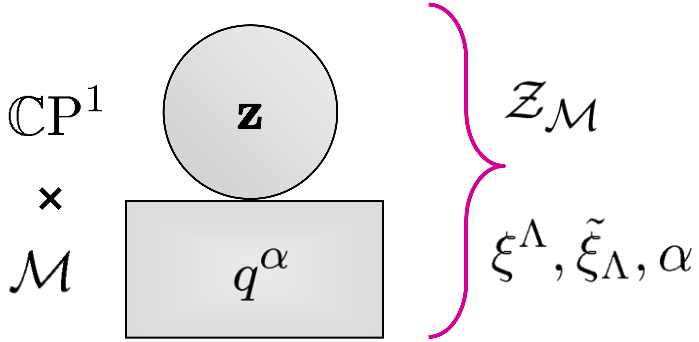

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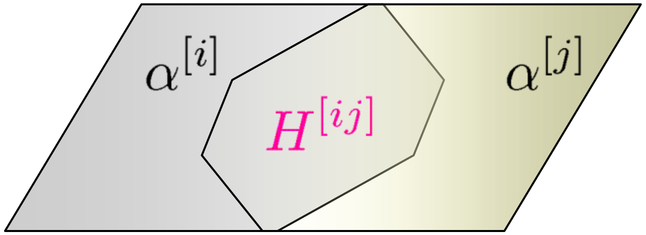
The main ingredient:

transition functions between different patches

$H^{[ij]}(\xi, \tilde\xi, \alpha)$



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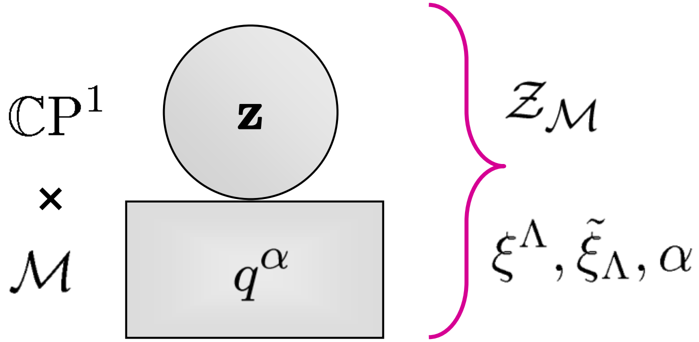

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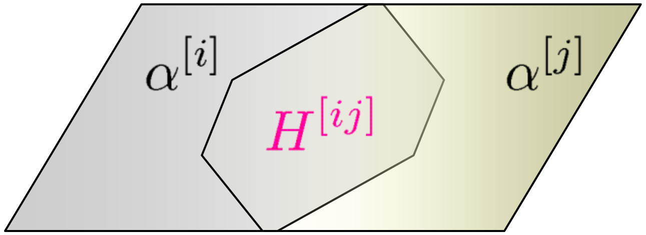

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twistor lines

$\xi^\Lambda_{[i]}(q^\alpha, \mathbf{z}),\ \ \tilde\xi^{[i]}_\Lambda(q^\alpha, \mathbf{z}),\ \ \alpha^{[i]}(q^\alpha, \mathbf{z})$

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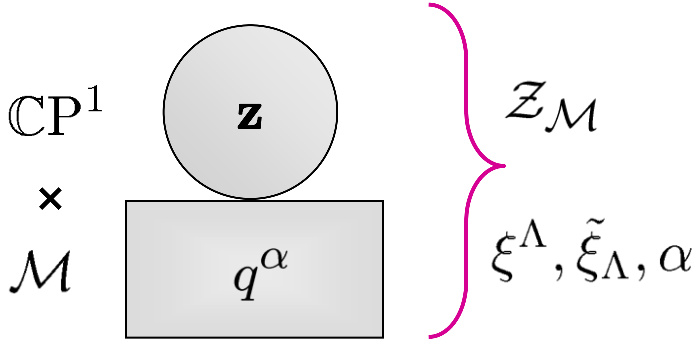

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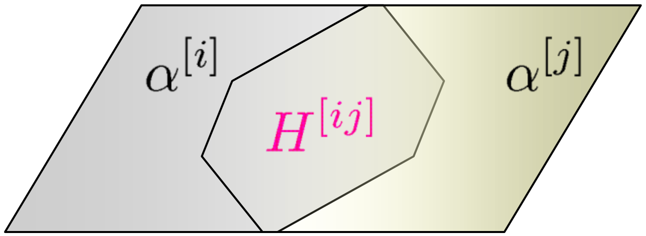

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metric on $\mathcal{M}$

Perturbative HM moduli space

HM moduli space at tree level

c-map →

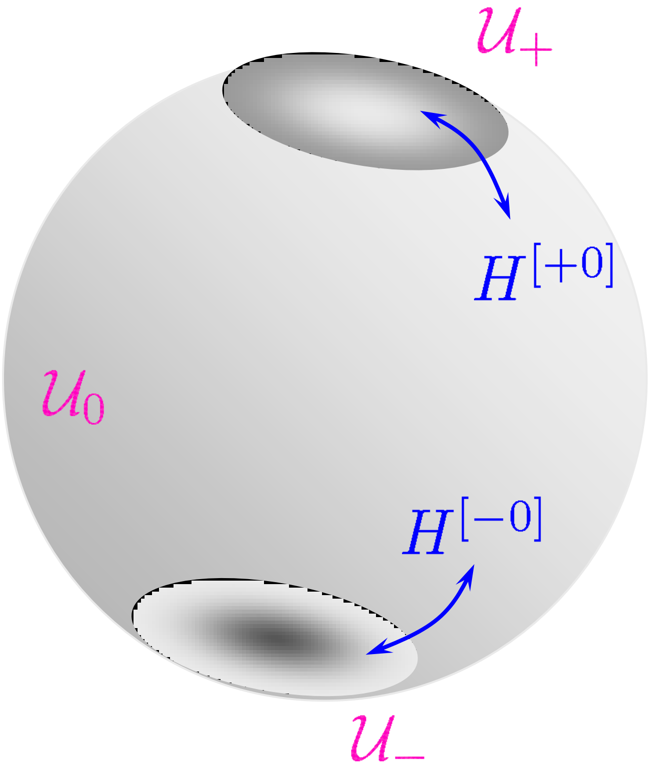
$$\boxed{H^{[+0]} = F(\xi) \qquad H^{[-0]} = \bar{F}(\xi)}$$

Twistor lines:

$$\begin{aligned} \xi^{\Lambda}_{[0]} &= \zeta^{\Lambda} + \frac{\tau_2}{2}\left(\mathbf{z}^{-1} z^{\Lambda} - \mathbf{z}\,\bar{z}^{\Lambda}\right) \\ \tilde{\xi}^{[0]}_{\Lambda} &= \tilde{\zeta}_{\Lambda} + \frac{\tau_2}{2}\left(\mathbf{z}^{-1} F_{\Lambda}(z) - \mathbf{z}\,\bar{F}_{\Lambda}(\bar{z})\right) \\ \alpha^{[0]} &= \sigma + \frac{\tau_2}{2}\left(\mathbf{z}^{-1} W(z) - \mathbf{z}\,\bar{W}(\bar{z})\right) \end{aligned}$$

where $W(z) \equiv F_{\Lambda}(z)\zeta^{\Lambda} - z^{\Lambda}\tilde{\zeta}_{\Lambda}$

- respect symplectic invariance, SL(2,$\mathbb{Z}$)-duality and mirror symmetry
- reproduce the known HM metric



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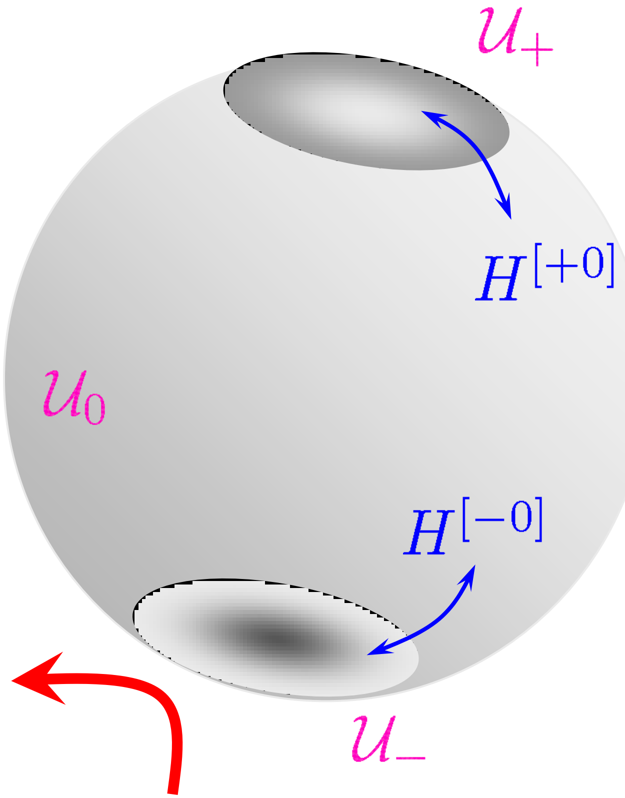

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One-loop correction

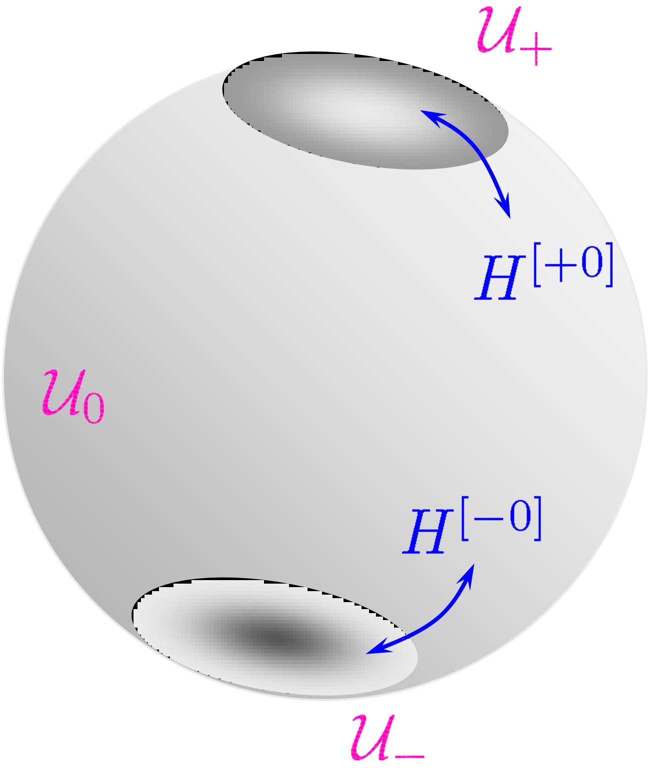
- appears as a singular boundary condition for twistor lines
- determined by the Euler number

Non-perturbative HM moduli space

(Type IIA picture)

Every D2 brane has the charge $\gamma = (q_\Lambda, p^\Lambda)$

(it wraps the cycle $q_\Lambda \gamma^\Lambda + p^\Lambda \gamma_\Lambda$)



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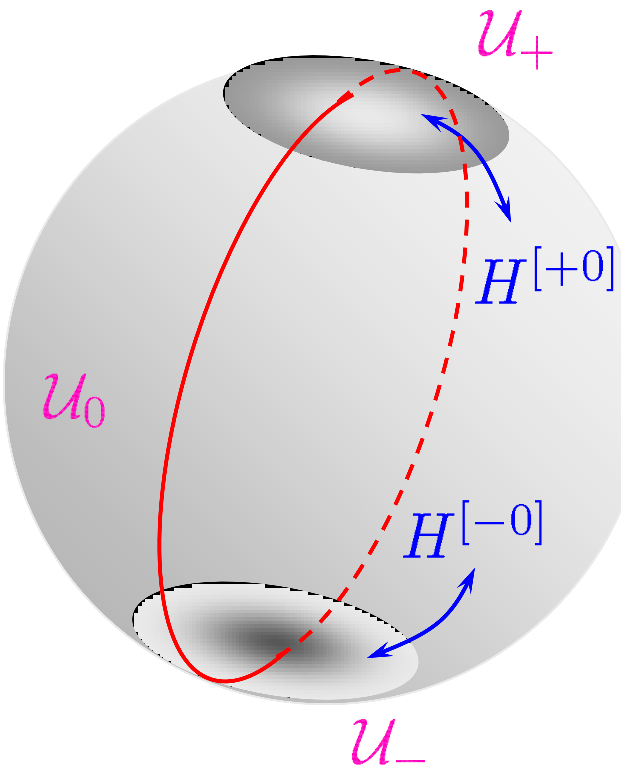
$Z_\gamma(z) = q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z)$ *— central charge*



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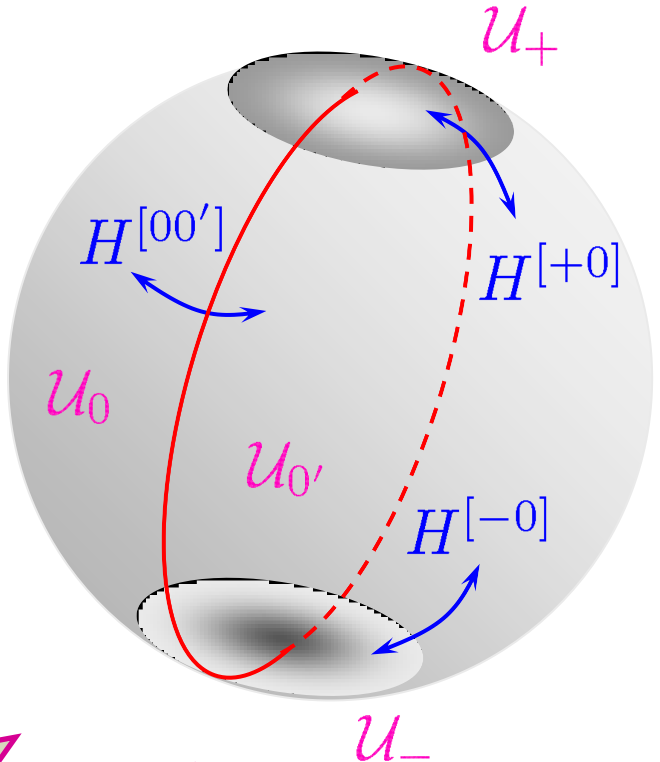

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One associates a discontinuity with each ray:

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$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

n_γ — generalized Donaldson-Thomas invariants satisfying Kontsevich-Soibelman wall-crossing formula

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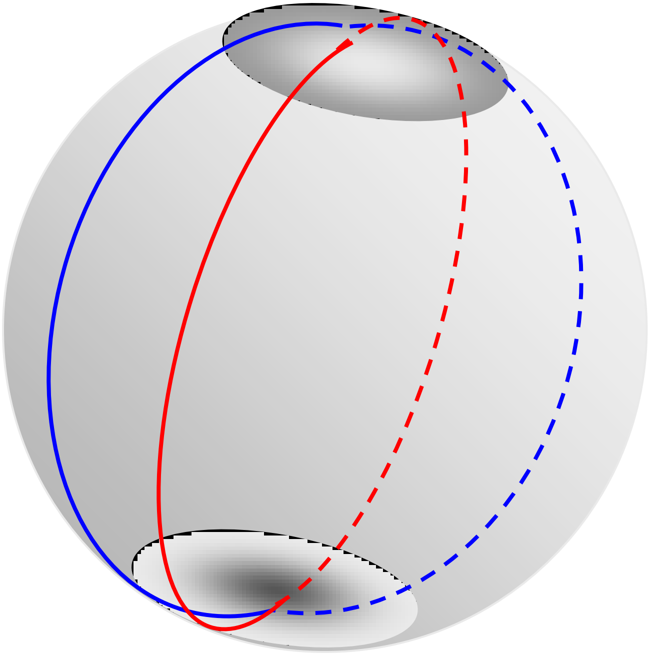

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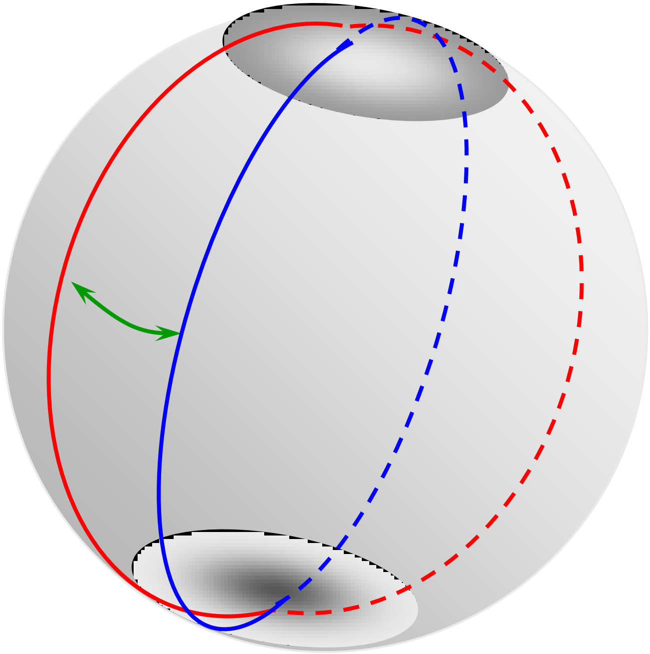

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$$\xi^0 \mapsto \frac{a\xi^0 + b}{c\xi^0 + d} \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0 + d}$$

$$\tilde{\xi}_a \mapsto \tilde{\xi}_a + \frac{\mathrm{i}c}{4(c\xi^0 + d)} \kappa_{abc}\xi^b\xi^c$$

$$\begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} + \begin{matrix} \text{non-linear} \\ \text{terms} \end{matrix}$$

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Find a map $q^\alpha_{\mathrm{IIA}} = \mathbf{m}[q^\alpha_{\mathrm{IIB}}]$ and
a *holomorphic* action of SL(2,$\mathbb{Z}$)
on the twistor space such that:

$$(\Xi(q^\alpha_{IIA}, \mathbf{z}) = (\xi^\Lambda, \tilde\xi_\Lambda, \alpha))$$

$$\mathbf{m} \xrightarrow[\alpha'\to 0,\, g_s\to 0]{} \mathbf{m}_{\mathrm{cl}}$$

$$\underbrace{g\cdot\Xi}_{\text{to be found}} = \Xi\left(\underbrace{\mathbf{m}}_{\text{to be found}}[\underbrace{g\cdot q^\alpha_{\mathrm{IIB}}}_{\text{known}}], \underbrace{g\cdot\mathbf{z}}_{\text{known}}\right)$$

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$$\tilde\xi_a \mapsto \tilde\xi_a + \frac{\mathrm{i}c}{4(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c$$

$$\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} \mapsto \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} + \begin{matrix}\text{non-linear}\\ \text{terms}\end{matrix}$$

Find a map $q^\alpha_{\mathrm{IIA}} = \mathbf{m}[q^\alpha_{\mathrm{IIB}}]$ and a *holomorphic* action of SL(2,$\mathbb{Z}$) on the twistor space such that:

$$(\Xi(q^\alpha_{IIA},\mathbf{z}) = (\xi^\Lambda,\tilde\xi_\Lambda,\alpha))$$

$$\mathbf{m} \xrightarrow[\alpha'\to 0,\, g_s\to 0]{} \mathbf{m}_{\mathrm{cl}}$$

$$\underbrace{g\cdot\Xi} = \Xi\left(\underbrace{\mathbf{m}}[\underbrace{g\cdot q^\alpha_{\mathrm{IIB}}}], \underbrace{g\cdot\mathbf{z}}\right)$$

to be found

known

One can do this by restricting to α', D(-1) and D1 corrections

As a result, the consistency condition crucially simplifies

The results

Instanton corrections to the mirror map

$$\tilde{\zeta}_a^{\rm inst} = \frac{1}{8\pi^2}\sum_{q_a\geq 0} n_{q_a}^{(0)} q_a \sum_{\substack{n\in\mathbb{Z}\\ m\neq 0}} \frac{m\tau_1+n}{m|m\tau+n|^2}\,\mathrm{e}^{-S_{m,n,q_a}}$$

$$\tilde{\zeta}_0^{\rm inst} = -\frac{\mathrm{i}}{16\pi^3}\sum_{q_a\geq 0} n_{q_a}^{(0)} \sum_{\substack{n\in\mathbb{Z}\\ m\neq 0}} \left[\frac{(m\tau_1+n)^2}{|m\tau+n|^3} + 2\pi q_a\left(t^a - \mathrm{i}b^a\,\frac{m\tau_1+n}{|m\tau+n|}\right)\right]\frac{\mathrm{e}^{-S_{m,n,q_a}}}{m|m\tau+n|}$$

$$\begin{aligned}\sigma^{\rm inst} = {} & \tau_1\tilde{\zeta}_0^{\rm inst} - (c^a - \tau_1 b^a)\,\tilde{\zeta}_a^{\rm inst} + \frac{\mathrm{i}\tau_2^2}{8\pi^2}\sum_{q_a\geq 0} n_{q_a}^{(0)} q_a t^a \sum_{n\neq 0}\frac{\mathrm{e}^{-S_{0,n,q_a}}}{n|n|} \\ & -\frac{\mathrm{i}}{8\pi^3}\sum_{q_a\geq 0} n_{q_a}^{(0)} \sum_{\substack{n\in\mathbb{Z}\\ m\neq 0}} \left(2 - \frac{(m\tau_1+n)^2}{|m\tau+n|^2}\right)\frac{(m\tau_1+n)\mathrm{e}^{-S_{m,n,q_a}}}{m^2|m\tau+n|^2}\end{aligned}$$

$$S_{m,n,q_a} = 2\pi q_a(|m\tau+n|t^a - \mathrm{i}mc^a - \mathrm{i}nb^a)$$

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The holomorphic action of SL(2,$\mathbb{Z}$) on the twistor space does not receive any corrections!

Gaiotto,Moore,Neitzke ’08

Instantons and TBA

$$\Xi_\gamma = q_\Lambda \xi^\Lambda_{[\gamma]} - p^\Lambda \tilde\xi_\Lambda^{[\gamma]}$$
$$\Theta_\gamma = q_\Lambda \zeta^\Lambda - p^\Lambda \tilde\zeta_\Lambda$$

Equation for the twistor lines:

$$\Xi_{\gamma_a}(\mathbf{z}) = \Theta_{\gamma_a} + \frac{\tau_2}{2}\left(\mathbf{z}^{-1} Z_{\gamma_a} - \mathbf{z}\bar Z_{\gamma_a}\right) + \frac{1}{8\pi^2}\sum_{b\neq a} n_{\gamma_b}\,\langle\gamma_a,\gamma_b\rangle \int_{\ell_{\gamma_b}} \frac{\mathrm{d}\mathbf{z}'}{\mathbf{z}'}\,\frac{\mathbf{z}+\mathbf{z}'}{\mathbf{z}-\mathbf{z}'}\,\log\left(1 - e^{-2\pi\mathrm{i}\Xi_{\gamma_b}(\mathbf{z}')}\right)$$

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$\beta m_a = 2\pi\tau_2|Z_{\gamma_a}|$

mass parameters

$\beta\mu_a = -2\pi\mathrm{i}\Theta_b + \pi\mathrm{i}$

chemical potentials

$\phi_{ab}(\theta) = -\frac{\mathrm{i}}{2}\langle\gamma_a,\gamma_b\rangle \frac{e^\theta + e^{\mathrm{i}(\theta_b-\theta_a)}}{e^\theta - e^{\mathrm{i}(\theta_b-\theta_a)}}$

kernel

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Thermodynamic Bethe Ansatz equations

$$\epsilon_a(\theta) = m_a\beta\cosh\theta - \frac{1}{2\pi}\sum_b n_{\gamma_b}\int_{-\infty}^{\infty}\mathrm{d}\theta'\,\phi_{ab}(\theta-\theta')\,\log\left(1+e^{\beta\mu_b-\epsilon_b(\theta')}\right)$$

Gaiotto,Moore,Neitzke '08

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Is it a formal analogy or something deep?

Potentials, free energy and YY-functional

Gauge theory:

Integrable model:

(non-perturb. symplectic
invariant part of)
the Kähler potential

=

Yang-Yang functional
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potential for QK spaces — *contact potential* giving Kähler potential on $\mathcal{Z}_{\mathcal{M}}$

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Without NS5 contributions it is globally defined and $\Phi^{[i]}(q^\alpha,\mathbf{z})=\phi(q^\alpha)\sim\log g_{(4)}^{-2}$

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S-matrix and integrability

$$\phi_{ab}(\theta) = -\mathrm{i}\frac{\partial \log S_{ab}}{\partial \theta} \implies$$

S-matrix

$$S_{ab}(\theta) = \left[\sinh\left(\frac{1}{2}\left(\theta + \mathrm{i}(\theta_a - \theta_b)\right)\right)\right]^{\langle \gamma_a, \gamma_b \rangle}$$

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Conditions on integrable S-matrices:

- Lorentz invariance
- Zamolodchikov algebra $S_{ab}(\theta)S_{ba}(-\theta) = 1$
- crossing symmetry $S_{b\bar{a}}(\pi\mathrm{i} - \theta) = S_{ab}(\theta)$
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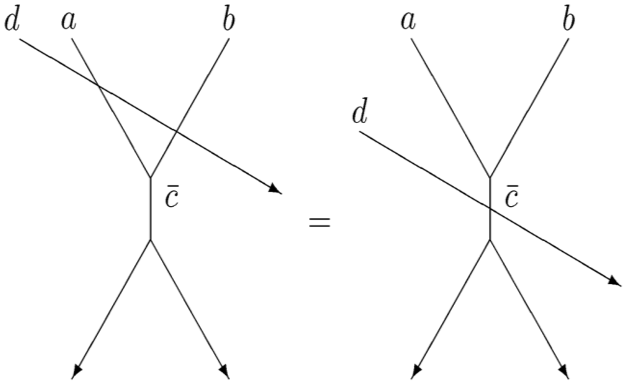

1. *there should be a bound state with*

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We have $u^c_{ab} = \theta_b - \theta_a$ *and the bound state carries the charge* $\gamma_{\bar{c}} = \gamma_a + \gamma_b$

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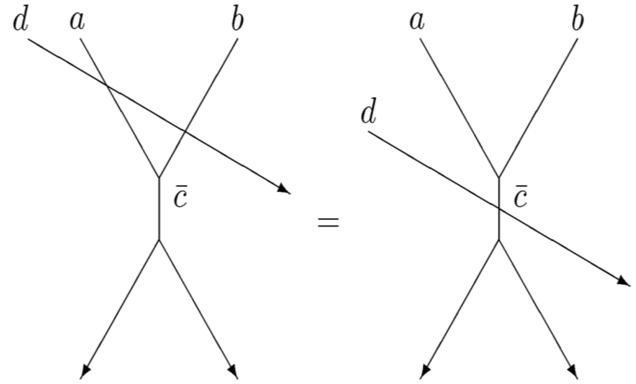

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All conditions on integrability are satisfied!

Conclusions

Results

- Twistor description of all D-instanton corrections to the HM moduli space
- Generalized mirror map including worldsheet, D(-1) D1-instanton corrections.

As a by-product: $SL(2,\mathbb{Z})$ transformations on the twistor space

- Relation to integrability

Problems

- To find corrections coming from D3 and D5-insantons

For this one needs the complete non-perturbative mirror map

- To find NS5-brane instanton corrections

See Pioline's talk