Einstein Gravity from A Matrix Integral

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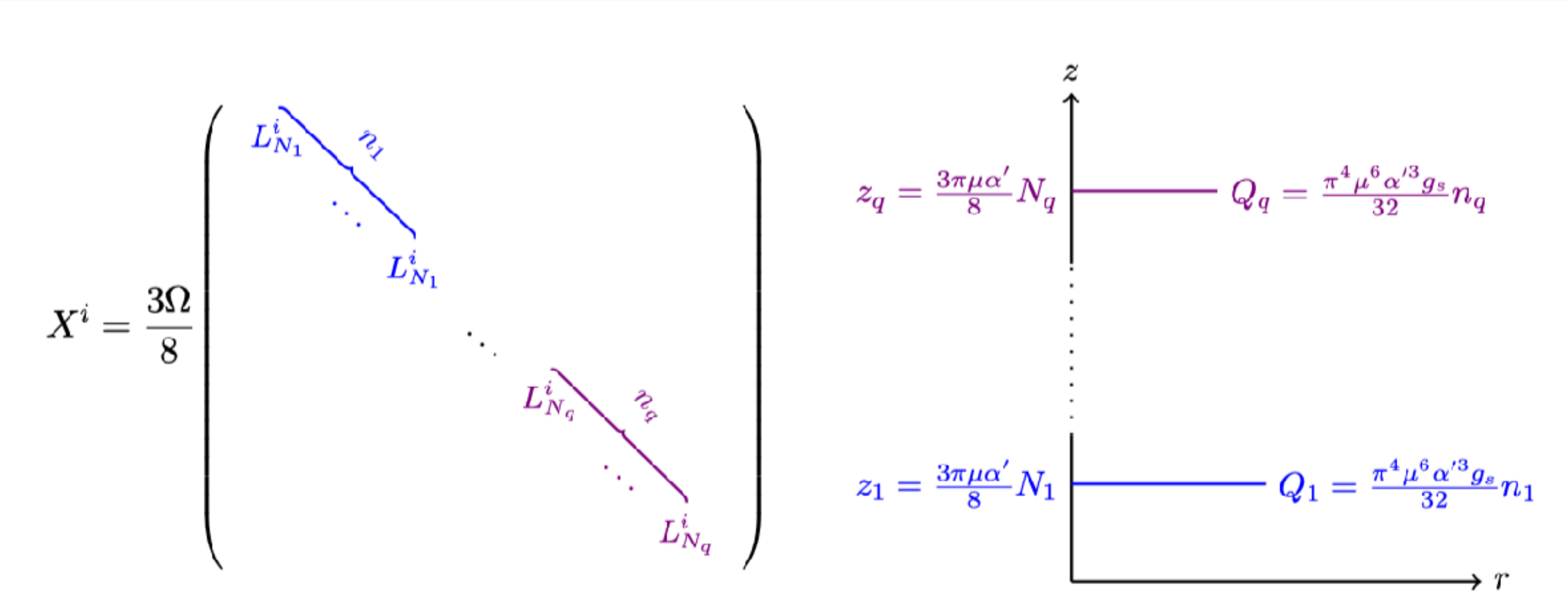
Motivation

- What is the simplest model of holography? Can we see the cartoon of emergence?
  - Matrix QM instead of QFT? Still too hard… [Komatsu, Martina, Penedones, Suchel, Vuignier, XZ ’24]
  - Matrix integrals are “technically simplest” (but conceptual challenges)
- Main result: duality between a matrix integral and a set of IIB SUGRA solutions



Outline

1. Review of IKKT matrix model and its mass deformation
2. Supersymmetry localisation
3. Gravitational dual
4. The matching
5. Summary & Outlook

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The IKKT matrix model

- The action (everything dimensionless, only **Euclidean** in this talk)

$$S_{\mathrm{IKKT}} = -\,\mathrm{Tr}\left[\frac{1}{4}\sum_{I,J=1}^{10}[X_I, X_J]^2 + \frac{i}{2}\sum_{\alpha,\beta=1}^{32}\sum_{I=1}^{10}\psi_\alpha\left(\mathcal{C}\Gamma^I\right)_{\alpha\beta}[X_I, \psi_\beta]\right] \qquad Z_{\mathrm{IKKT}} = \int \prod_{I=1}^{10} dX_I \prod_{\alpha=1}^{16} d\psi_\alpha e^{-S_{\mathrm{IKKT}}}$$

[Ishibashi, Kawai, Kitazawa, Tsuchiya ’97]

- X_I, ψ_α: $N \times N$ hermitian matrices
- Symmetry: $SU(N)$ gauge sym, $SO(10)$ global sym, $\mathcal{N} = 16$ susy
- From dimensional reduction of 10d $\mathcal{N} = 1$ SYM to 0+0d; low energy dynamics of N D-instantons.

The IKKT matrix model - what's known

- The partition function

$$Z_{\text{IKKT}} = \frac{(2\pi)^{(10N+11)(N-1)/2}}{\sqrt{N}\prod_{k=1}^{N-1}(k!)}\sigma_{-2}(N) \qquad \sigma_{-2}(N) = \sum_{m|N}\frac{1}{m^2}$$

[Green, Gutperle '97]
[Moore, Nekrasov, Shatashivili '98]
[Krauth, Nicolai, Staudacher '98]
[Austing, Wheater '01, '03]

- The gravity dual (backreacted geometry of N coincident D-instantons)

$$ds_E^2 = dr^2 + r^2 d\Omega_9^2$$

$$e^{\phi} = e^{\phi_\infty} + \frac{c}{r^8} \qquad \chi = -e^{-\phi} + (\chi_\infty + e^{-\phi_\infty})$$

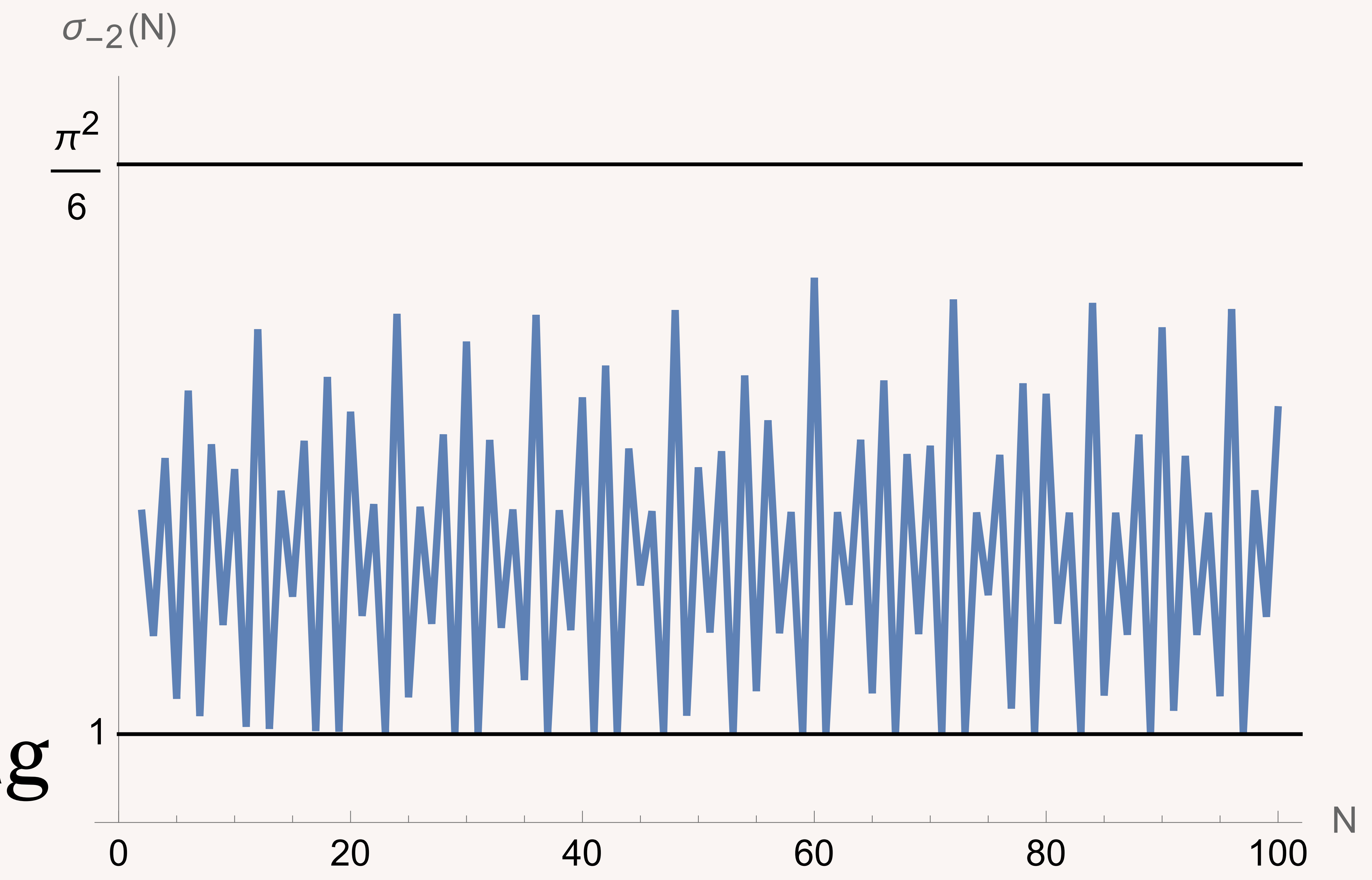
[Gibbons, Green, Perry '95]
[Ooguri, Skenderis '98]

- But the duality is *not* verified

The IKKT matrix model - conceptual issues

- The partition function has no large N limit

  [Krauth, Nicolai, Staudacher ’98]



- No time, no energy scale, no (meaningful) coupling
  - How to go to low energy take the decoupling limit?
  - How does time emerge?
  - How to realise the Hilbert space and physical states?

The polarised IKKT matrix model

- The action: $S_{\text{polarised}} = S_{\text{IKKT}} + S_\Omega$ [Bonelli '02; Hartnoll, Liu '24]

$$S_\Omega = \mathrm{Tr}\left[\frac{3\Omega^2}{4^3}\sum_{i=1}^{3} X_i X_i + \frac{\Omega^2}{4^3}\sum_{p=4}^{10} X_p X_p + i\frac{\Omega}{3}\epsilon_{ijk}X_i X_j X_k - \frac{\Omega}{8}\psi_\alpha \left(\mathcal{C}\Gamma^{123}\right)_{\alpha\beta}\psi_\beta\right]$$

unique mass deformation [Martina (to appear)]

- Symmetry: $SU(N)$, $SO(3)\times SO(7)$, $\mathcal{N} = 16$ (same amount as IKKT)
- Advantages over IKKT:
  - Uplifts the flat direction - IKKT "in a box", tuneable dim.-less coupling $\Omega \to \begin{cases} \infty : & \text{weak coupling} \\ 0 : & \text{strong coupling} \end{cases}$
  - Well-defined large N limit; duality with gravity established
- Remaining issues:
  - timeless, no energy scale, no notion of Hilbert space...

The polarised IKKT matrix model

- Classical vacua (*local* minima of bosonic potential):

$$X_p = 0, \quad X_i = \frac{3}{8}\Omega L_i, \quad [L_i, L_j] = i\epsilon_{ijk}L_k, \qquad p = 4,5,\ldots,10, \quad i = 1,2,3$$

- L_i: N-dimensional representation of $SU(2)$ generators (reducible in general)

$$L^i = \begin{pmatrix} \mathbb{1}_{n_1} \otimes L^i_{N_1} & & & & \\ & \ddots & & & \\ & & \mathbb{1}_{n_s} \otimes L^i_{N_s} & & \\ & & & \ddots & \\ & & & & \mathbb{1}_{n_q} \otimes L^i_{N_q} \end{pmatrix}$$

- Total number = integer partitions of N
- Non-degenerate: The N-dim. irrep. vacuum dominates (the lowest potential)

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Supersymmetry localisation - Idea

- Similar to IKKT, $Z_{\text{polarised}}$ can be computed by supersymmetry localisation
- Idea:
  - Select one supercharge Q, deform the action by Q-exact quantity $S \to S + tQV$
    - We choose to preserve $\phi = X_3 - iX_{10}$: $Q\phi = 0$
  - Z is invariant $\forall t$, send $t \to \infty$, path integral $\int [D\Phi] e^{-S-tQV}$ localises to saddles of QV
  - Correlators $\langle f(\phi) \rangle$ also computable
- Similar computations for BMN were done in [Asano, Ishiki, Okada, Shimasaki '12]

Supersymmetry localisation - Result

- The saddle points of QV: $\boxed{X_i = \frac{3}{8}\Omega L_i\,,}$ $\quad X_{10} = M\,, \quad [M, L_i] = 0\,, \quad i = 1,2,3$

Same as classical vacua

$$L^i = \begin{pmatrix} \mathbb{1}_{n_1} \otimes L^i_{N_1} & & & & \\ & \ddots & & & \\ & & \mathbb{1}_{n_s} \otimes L^i_{N_s} & & \\ & & & \ddots & \\ & & & & \mathbb{1}_{n_q} \otimes L^i_{N_q} \end{pmatrix} \qquad M = \begin{pmatrix} M_1 \otimes \mathbb{1}_{N_1} & & & & \\ & \ddots & & & \\ & & M_s \otimes \mathbb{1}_{N_s} & & \\ & & & \ddots & \\ & & & & M_q \otimes \mathbb{1}_{N_q} \end{pmatrix}$$

- Other matrices vanish

- Henceforth, s: irrep of $\mathfrak{su}(2)$; N_s: dimension of type-s irrep; n_s: multiplicity of type-s irrep
- $\sum_{s=1}^{q} n_s N_s = N$

Supersymmetry localisation - Result

- The partition function: $Z = \sum_{\mathcal{R}} Z_{\mathcal{R}}\,,\quad \mathcal{R} = \text{label of saddle points } (N\text{-dim. rep. of } \mathfrak{su}(2))$

$$Z_{\mathcal{R}} = C_{\mathcal{R}} e^{\frac{9\Omega^4}{2^{15}}\sum_s n_s (N_s^3 - N_s)} \int \left(\prod_s \prod_{i=1}^{n_s} dm_{si} \right) Z_{1-\text{loop}} \exp\left(-\frac{3\Omega^4}{2^7} \sum_s \sum_{i=1}^{n_s} N_s m_{si}^2 \right),$$

constant — Eigenvalues of $X^{10} = M$ — Gaussian

$$Z_{1-\text{loop}} = \prod_{(si,tj)} \prod_{J=\frac{|N_s-N_t|}{2}}^{\frac{N_s+N_t}{2}-1} \frac{[(2+3J)^2 + (8(m_{si}-m_{tj}))^2]^3 [(3J)^2 + (8(m_{si}-m_{tj}))^2]}{[(1+3J)^2 + (8(m_{si}-m_{tj}))^2]^3 [(3+3J)^2 + (8(m_{si}-m_{tj}))^2]}$$

generalised Vandermonde determinant; two-body interaction potential

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The gravity dual - Idea

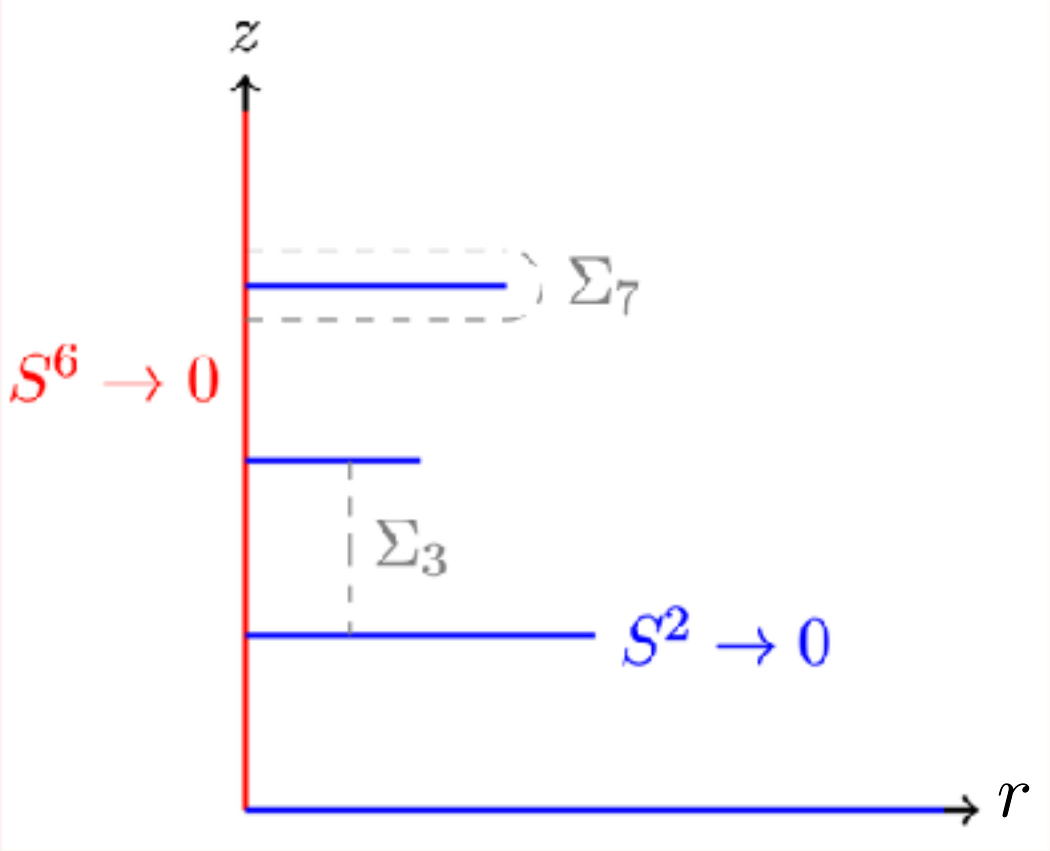
- **Goal**: Construct the IIB SUGRA solution dual to the polarised IKKT model
- **Idea**: Be guided by symmetry.
  - The bosonic symmetry is $SO(7) \times SO(3)$.
  - The IIB SUGRA dual to 5d SCFT with $SO(2,5) \times SO(3)$ isometry has been constructed. [D'Hoker, Gutperle, Karch, Uhlemann '16, '17; Legramandi, Neunez '21]
    - Exceptional Lie superalgebra F_4 is the unique 5d superconformal algebra.
  - The superalgebra of polarised IKKT should be a different real form of F_4.
- **Strategy**: Take appropriate analytic continuation of the above SUGRA solution.

The gravity dual - Result

- The metric: $$ds^2 = H^2(r,z)(dr^2 + dz^2) + R_2^2(r,z)d\Omega_2^2 + R_6^2(r,z)d\Omega_6^2$$

  - $S^2 \times S^6$ fibred over a 2D plane:
  - Boundaries in (r, z) plane: shrinking of S^2 and S^6

$R_6^3 R_2 \propto r^2 \partial_r V(r,z)$



$\times S^2 \times S^6$

- H, R_2, R_6 and e^ϕ, χ, B_2, C_2 all determined by $V(r, z)$ and a mass parameter μ.
  - Asymptotic $r, z \to \infty$: flat space with constant background flux $dC_2 = -i\mu dx^1 \wedge dx^2 \wedge dx^3$.
- By construction $V(r, z)$ has to satisfy the 4D Laplace equation (with axial symmetry):

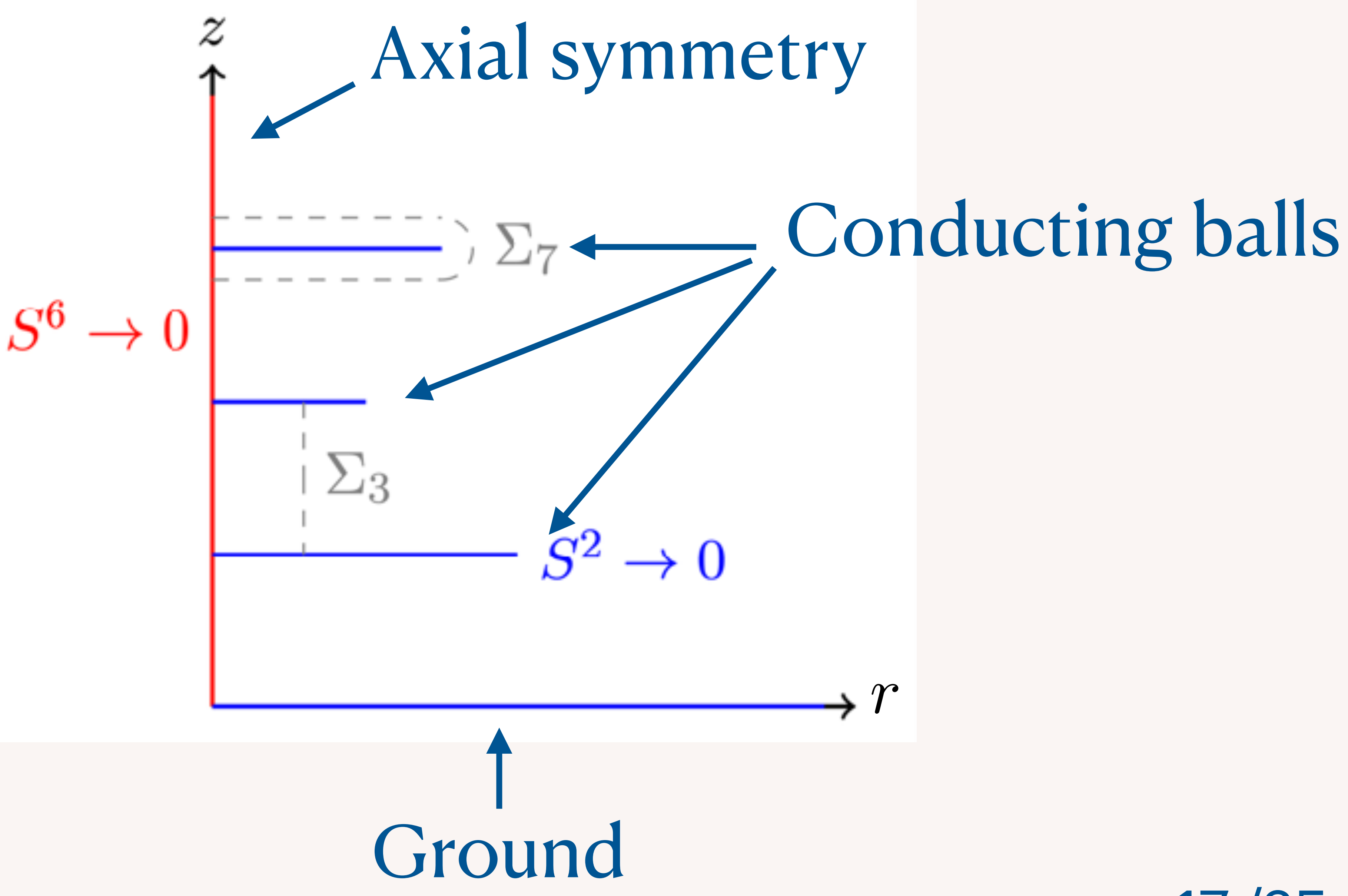
$$\partial_r^2 V + \frac{2}{r}\partial_r V + \partial_z^2 V = 0$$

$V(r, z)$ is an electrostatic potential

Gravity from electrostatics

- Imposing positivity and smoothness of the geometry, the problem of determining the back-reacted geometry becomes an **electrostatic problem**...
  - in 4D with axial sym. around z-axis (two angular directions are unphysical)
  - B.C. = conducting 3D balls at constant z + infinite ground
  - background potential $V_{bg} \propto \mu^5(zr^2 - z^3)$
- Similar to Lin-Maldacena geometry dual to BMN

[Lin, Maldacena '05]



Charge quantisation

- Dirac charge quantisation (electrostatic par. → IIB par.)

$$N_{D(-1)} \overset{\text{def}}{\propto} \frac{1}{g_s}\int_{S^9} \tilde{F}_9 \overset{\text{calc}}{\propto} P$$

dipole moment

$$N_{D1} \overset{\text{def}}{\propto} \frac{1}{g_s}\int_{\Sigma_7} \tilde{F}_7 \overset{\text{calc}}{\propto} Q$$

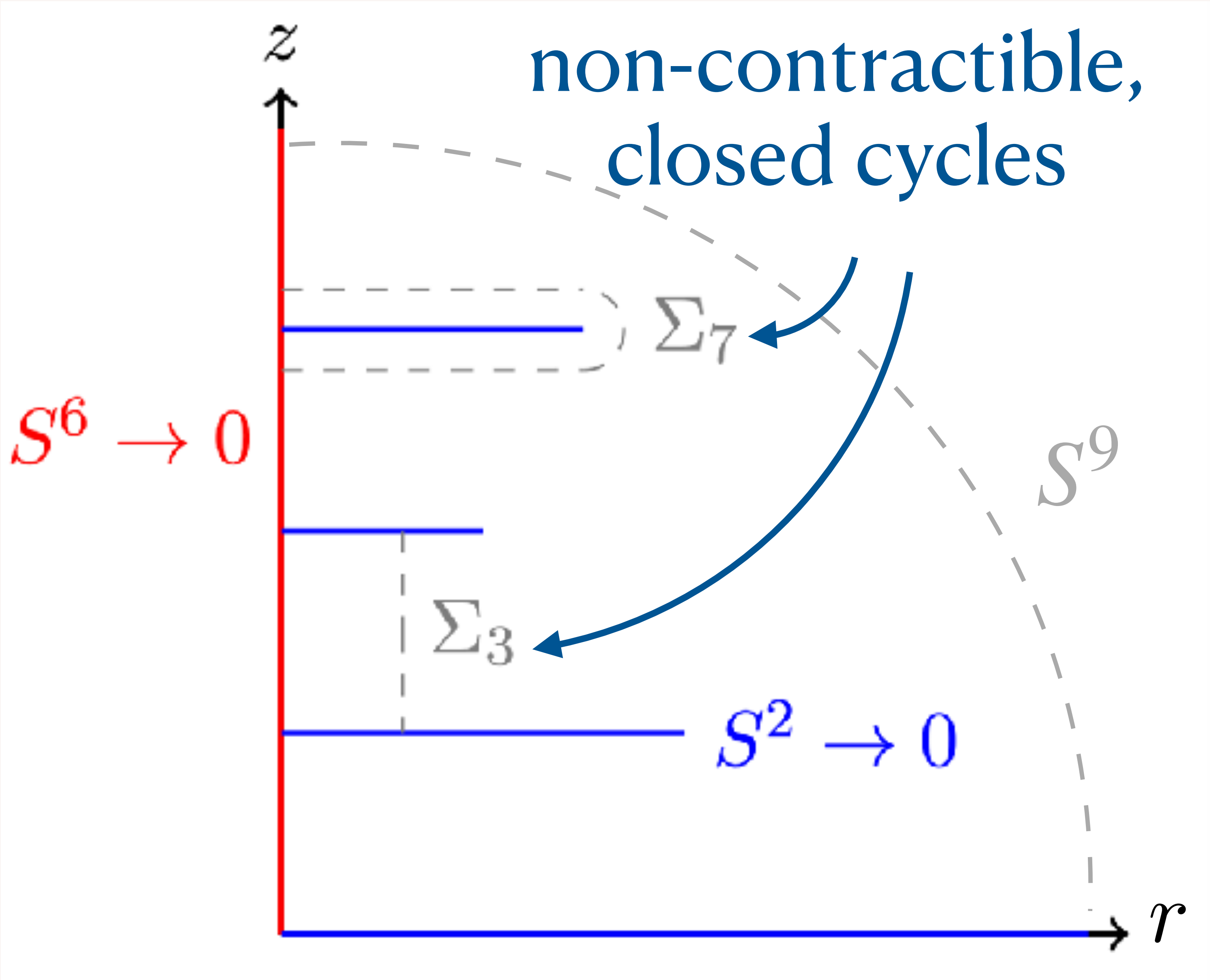
electric charge of ball

$$N_{F1} \overset{\text{def}}{\propto} \int_{\Sigma_7} \tilde{H}_7 \overset{\text{calc}}{=} 0$$

$$N_{NS5} \overset{\text{def}}{\propto} \int_{\Sigma_3} H_3 \overset{\text{calc}}{\propto} \Delta z$$

height difference

$$d\tilde{F}_9 = d\tilde{F}_7 = d\tilde{H}_7 = dH_3 = 0$$



- Radius of ball r_s is fixed by vanishing charge density at the tip $r_s \propto Q_s^{\#>0}$.

Outline

The matching

- From localisation: select a saddle point of QV labelled by $\mathcal{R}$ and rewrite

$$Z_{\mathcal{R}} = \mathcal{C}_{\mathcal{R}} \int \prod_{s,i} dm_{si} e^{-S_{\text{eff}}[\rho^{(s)}(x)]}$$

Eigenvalue density ($n_s \gg 1$): $\rho^{(s)}(x) = \sum_{i=1}^{n_s} \delta(x - m_{si})$

- At large N, the saddle point eqn $\delta S_{\text{eff}}[\rho^{(s)}(x)]/\delta\rho^{(s)}(x) = 0$ gives

$$\frac{3\Omega^4}{2^7} N_s x^2 - \mu_s + \frac{2}{3} \sum_{t+\text{images}} \int_{-x^{(t)}}^{x^{(t)}} dy \frac{(\frac{3}{2}|N_s - N_t|)^2 - (8(x-y))^2}{\left((\frac{3}{2}|N_s - N_t|)^2 + (8(x-y))^2\right)^2} \rho^{(t)}(y) = 0$$

Lagrange multiplier to impose

$$\int \rho^{(s)} = n_s$$

to match with gravity

The matching

- From localisation:

$$\frac{3\Omega^4}{2^7} N_s x^2 - \mu_s + \frac{2}{3} \sum_{t+\text{images}} \int_{-x^{(t)}}^{x^{(t)}} dy \frac{(\frac{3}{2}|N_s - N_t|)^2 - (8(x-y))^2}{\left((\frac{3}{2}|N_s - N_t|)^2 + (8(x-y))^2\right)^2} \rho^{(t)}(y) = 0$$

- From gravity/electrostatics: the Laplace eqn ⇔

elec. pot. on ball *s* —— $V_s = V_{bg}\left(r^2 z_s - z_s^3\right) + \sum_{t+\text{images}} V_{\text{ball},t}\left(r, z_s\right)$

$$\Downarrow$$

$$-V_s + V_{bg}(3r^2 z_s - z_s^3) + \sum_{t+\text{images}} \frac{1}{(2\pi)^2} \int_{-R_t}^{R_t} dr' \frac{(z_s - z_t)^2 - (r - r')^2}{((z_s - z_t)^2 + (r - r')^2)^2} f_t(r') = 0$$

radial charge density

The same form

- The dictionary:

emergent coordinate *r* —— $x = \frac{1}{2\pi\mu\alpha'} r\,, \qquad \rho^{(s)}(x) \propto f_s(r)$

$$P \propto N_{D(-1)} \leftrightarrow N\,, \quad z_s \propto \sum_{t \leq s} N_{NS5,t} \leftrightarrow N_s\,, \quad Q_s \propto N_{D1,s} \leftrightarrow n_s\,, \quad \mu \left(\frac{g_s}{(2\pi)^3 \alpha'^2}\right)^{-\frac{1}{4}} = \Omega$$

dimension of irrep_s (for N_s) multiplicity of irrep_s (for n_s)

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Summary

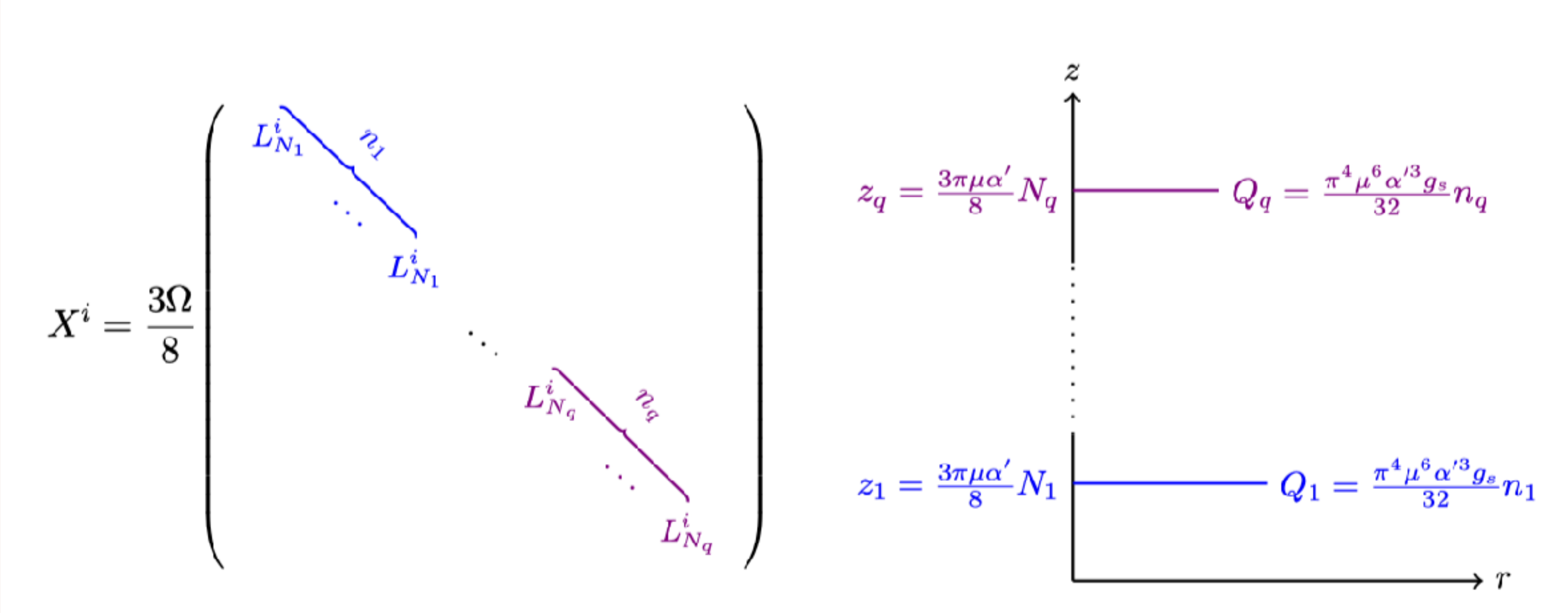
Polarised IKKT Matrix model

↓

Susy localisation

↓

Eigenvalue distribution



IIB SUGRA with F_4 SUSY

↓

Electrostatic analysis

↓

Electric charge distribution

Each saddle point $Z_{\mathcal{R}}$ corresponds to a different SUGRA solution.

$$P \propto N_{D(-1)} \leftrightarrow N\,, \quad z_s \propto \sum_{t\leq s} N_{NS5,t} \leftrightarrow N_s\,, \quad Q_s \propto N_{D1,s} \leftrightarrow n_s\,, \quad \mu\left(\frac{g_s}{(2\pi)^3\alpha'^2}\right)^{-\frac{1}{4}} = \Omega$$

dimension of irrep$_s$ (for N_s)

multiplicity of irrep$_s$ (for n_s)

Outlook

- To see the "cartoon": What are the correct matrix model observables for emergent geometry?
  - Pillon action: $$e^{-N S^{(k)}_{\text{Pillon}}[x^I]} \equiv \frac{1}{Z_N} \int [dX]_x e^{-S_{N+k}[X^I]}$$

$$X = \begin{pmatrix} x_{k\times k} & \bar{w}_{k\times N} \\ w_{N\times k} & \tilde{X}_{N\times N} \end{pmatrix} \qquad S^{(k)}_{\text{Pillon}}[x^I] \supset \{e^{\phi}, \chi, G_{MN}, B, C_2, C_4\}$$

[Ferrari '12,'13,'13…; Komatsu, Martina, Penedones, Vuignier, XZ (to appear)]

- Identifying the gravity dual to *Lorentzian* polarised IKKT ($dS_6 \times S^2$?)
- Emergence of time?

Thank you!

Supplementary

Relation with Z_{IKKT} - a surprise

- $S_{\text{polarised}}(\Omega = 0) = S_{\text{IKKT}}$, but $\lim_{\Omega \to 0} Z_{\text{polarised}} \neq Z_{\text{IKKT}}$
    - In the strong coupling $\Omega \to 0$, trivial rep. dominates:

$$Z_{\mathcal{R}} = O\left(\Omega^{2-2\sum_s n_s}\right) = \begin{cases} O\left(\Omega^{2-2N}\right), & \mathcal{R} = \text{trivial rep.} \\ O(1), & \mathcal{R} = N\text{-dim. irrep.} \end{cases}$$

    - Thus $Z_{\text{polarised}}$ diverges at $\Omega \to 0$; while$Z_{\text{IKKT}} \propto \sigma_{-2}(N)$.
- The fermonic mass term modifies the Pfaffian and the convergence of Z.
    - $\text{Pf}(X) \propto (\text{Tr}[X_I, X_J][X_J, X_K][X_K, X_I])^4$ vs $\text{Pf}(X) \propto \Omega^8 |X|^{16} \sim |X|^8$ ($N = 2$, diagonal modes)

      [Krauth, Nicolai, Staudacher '98]
- The path integral and the limit do not commute.

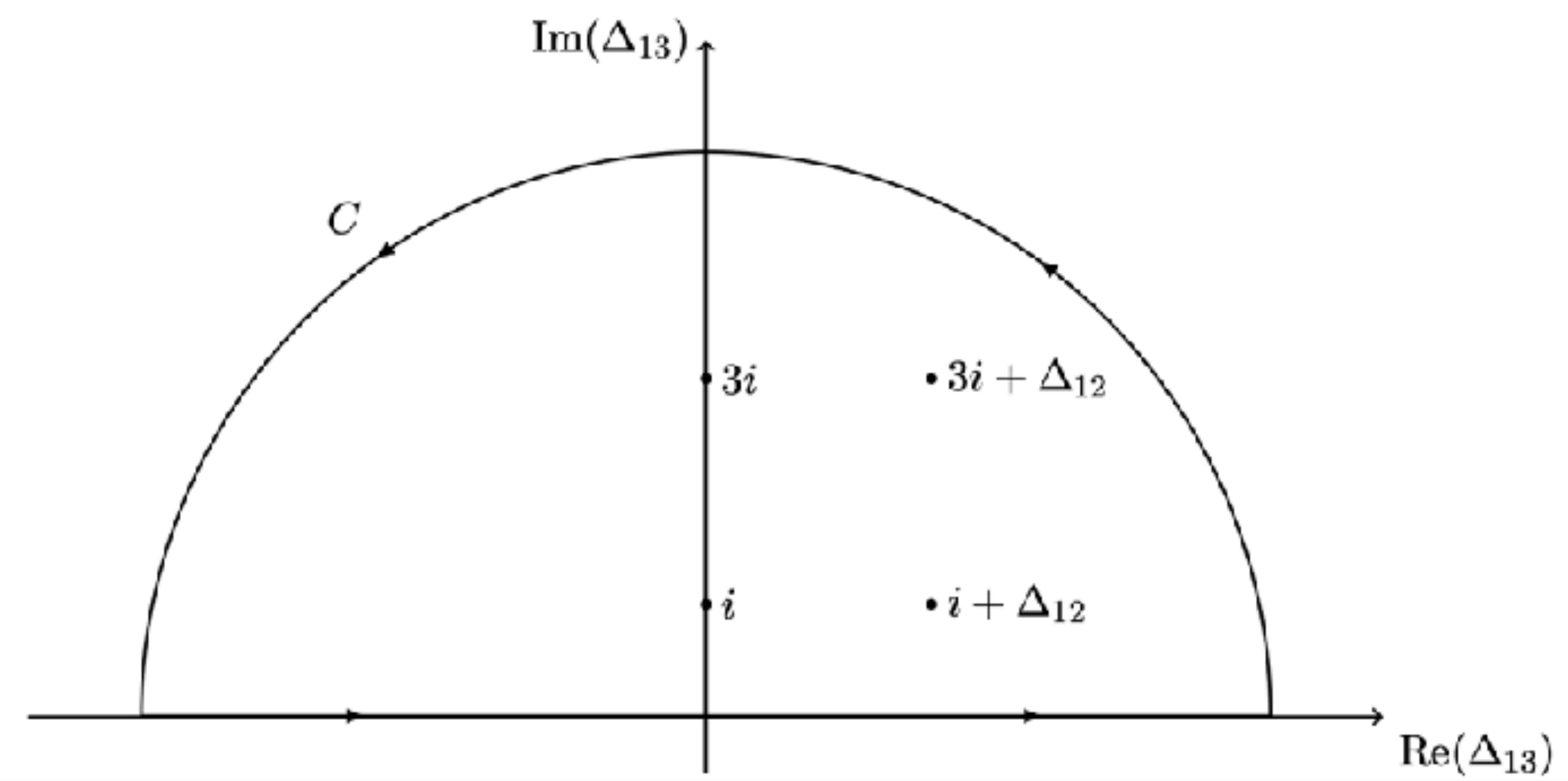
Relation with Z_{IKKT} - a conjecture

- We conjecture a prescription to "regularise" $Z_{\mathrm{polarised}}$, and $Z^{(\mathrm{reg})}_{\mathrm{polarised}} \overset{\Omega\to 0}{=} Z_{\mathrm{IKKT}}$

  1. Set $\Omega = 0$ before the m_{si} integral (no Gaussian factor)
  2. Deform the integral contour of m_{si} from $\mathbb{R}$ to UHP
  3. Neglect the arc @ ∞, only pick up the poles

  - Checked for $N = 2,3$ explicitly

- Proof for general N? Physical interpretation?



Fundamental objects in Euclidean IIB SUGRA

- The Euclidean IIB SUGRA action (bosonic part):

$$S^{(E)}_{\mathrm{IIB}} = \frac{1}{(2\pi)^7 \alpha'^4 g_s^2} \int d^{10}x \sqrt{g} \left[-R + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} e^{2\phi}(\nabla\chi)^2 + \frac{1}{12} e^{-\phi}(H_3)^2 + \frac{1}{12}(F_3)^2 \right] \qquad F_3 = dC_2 - \chi H_3$$

	NS-NS sector	R-R sector		
Electrically charged	F1 (Fundamental string)	**D(-1) (D-instanton)**	**D1**	D3
Magnetically charged	**NS5**	D7	D5	D3
Gauge field	B2 (Kalb-Ramond)	χ **(axion)**	C2	C4
Field strength	H3(=dB2)	dχ	dC2	F5(=dC4)

$SL(2,\mathbb{R})$ internal symmetry:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \tau = \chi + i e^{-\phi}$$

$$\begin{pmatrix} H_3 \\ dC_2 \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H_3 \\ dC_2 \end{pmatrix}$$

$$ad - bc = 1$$

χ and C2 are purely imaginary in Euclidean

The full form of back-reacted geometry

$$ds^2 = \frac{8}{\mu^{\frac{5}{2}}}\left(\frac{1}{3^3}\frac{\Delta\dot{V}}{(-V'')}\right)^{1/4}\left[\frac{(-V'')}{\dot{V}}(d\rho^2 + dz^2) + 3\rho d\Omega_6^2 + \frac{\rho(-V'')\dot{V}}{\Delta}d\Omega_2^2\right],$$

$$e^{\phi} = -\mu^3\frac{3\dot{V} + \rho V''}{\rho\sqrt{3\Delta\dot{V}(-V'')}},$$

$$\chi = -\frac{i}{\mu^3}\frac{3\dot{V}(V' + \rho\dot{V}') + \rho V'V''}{3\dot{V} + \rho V''},$$

$$H_3 = dB_2, \qquad B_2 = -\frac{8}{3\mu}\left(z - \frac{\rho\dot{V}\dot{V}'}{\Delta}\right)\wedge d\Omega_2$$

$$F_3 = dC_2 - \chi H_3, \qquad C_2 = i\frac{8}{3\mu^4}\left(V - \rho\frac{\dot{V}}{\Delta}\left(V'\dot{V}' + 3\dot{V}(-V'')\right)\right)\wedge d\Omega_2,$$

$$\Delta \equiv 3\dot{V}V'' + \rho(\dot{V}'^2 + V''^2).$$

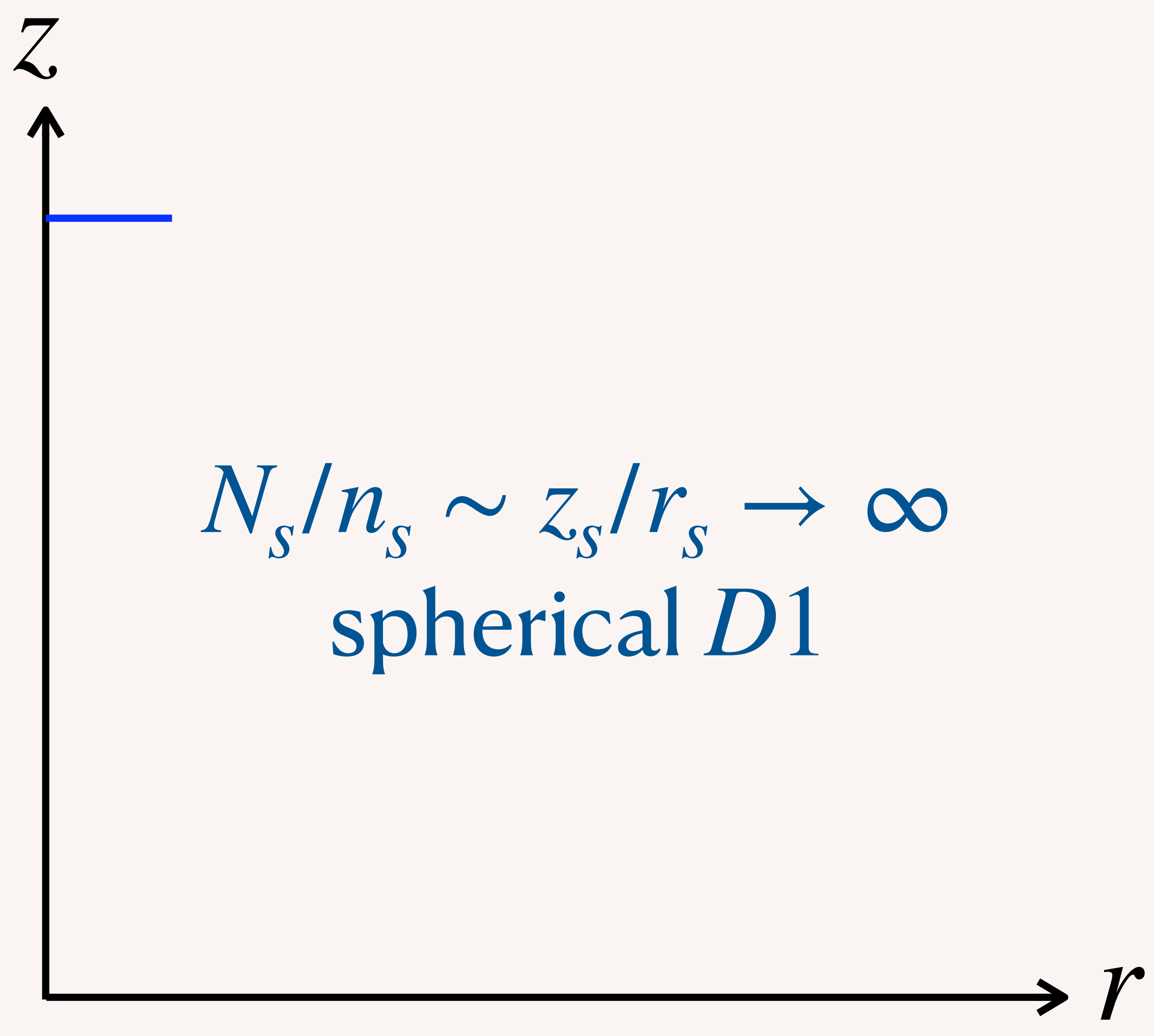
$$[r] = [z] = [M]^{-1}$$

$$[\mu] = [M]$$

$$[V] = [M]^2$$

Two limits

$r_s \propto Q_s^{\#>0}$

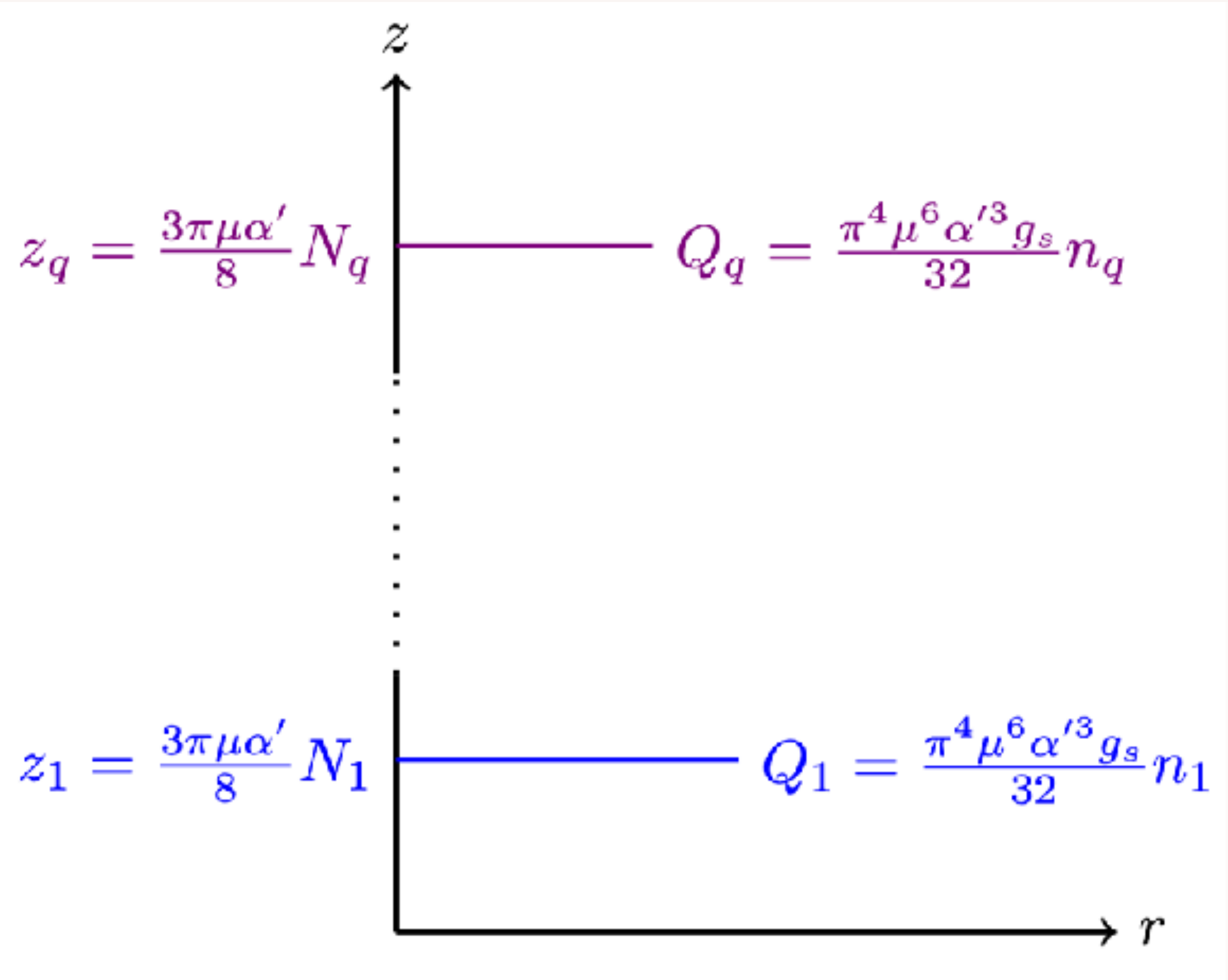


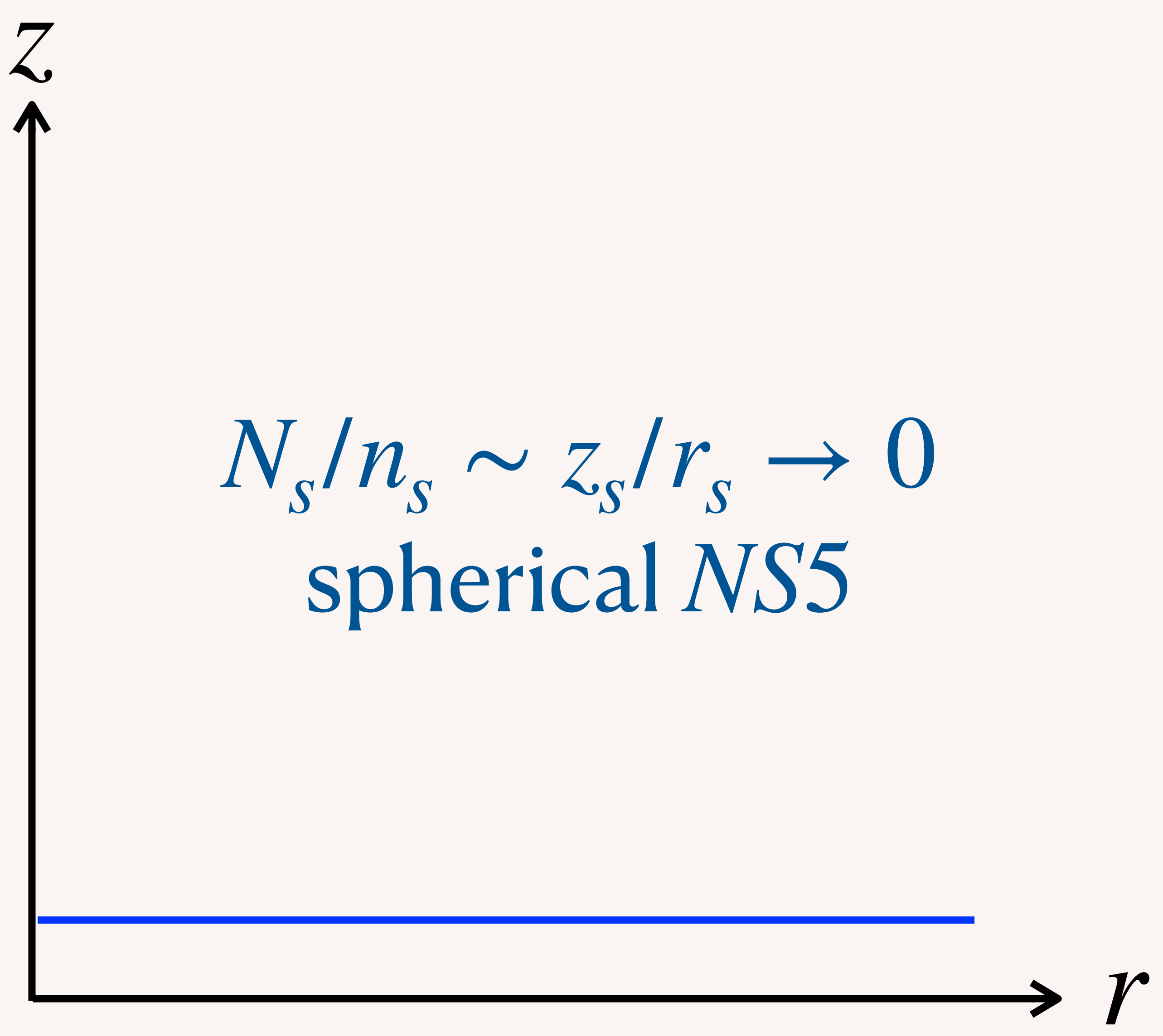
$N_s/n_s \sim z_s/r_s \to \infty$

spherical $D1$

N-dim. irrep.

(SUGRA invalid here)





$N_s/n_s \sim z_s/r_s \to 0$

spherical $NS5$

Trivial rep.

Free energy matching issue

- SUGRA on-shell action: boundary term?

$$S_E = \frac{1}{2\kappa^2}\int d\left(-\frac{1}{4}M_{ij}\mathcal{C}_2^i \wedge *\mathcal{F}_3^j\right)$$