Completeness of the Bootstrap

French Strings Meeting

Tours, June 3 2025

What is Quantum Field Theory?

- integral over "fields" $\Phi(x)$ varying in space (scalars, fermions, gauge fields, discrete d.o.f., ...)
- machine for calculating correlation functions

$$\langle \ldots \rangle = \int \ldots \, d\mu(\Phi)$$

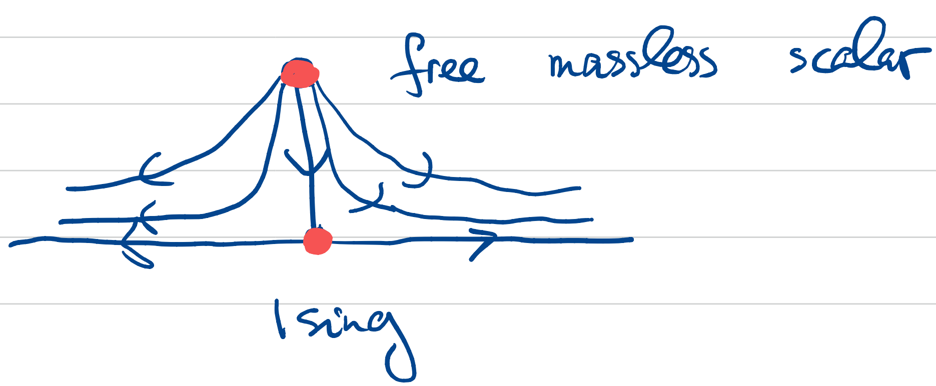
- $\Phi(x) \sim$ "functions"

$$d\mu(\Phi) \sim \text{"} \prod_x d\mu_x(\Phi) \text{"} \quad (\text{locality})$$

- local operators $\mathcal{O}_i(x)$ = built out of $\Phi(x+\epsilon)$
- can deform the action

$$d\mu' = \exp\left\{ \int \sum_i g_i \, \mathcal{O}_i(x) \, dx \right\} d\mu$$

- space of QFTs Ω = space of couplings $(g_i)_{i=1}^{\infty}$
- rescaling $x \to \lambda x$: $\mathbb{R}_{>0} \times \Omega \to \Omega$
- fixed points = conformal field theories



Goal: classify CFTs

Problems:
- no precise global notion of Ω.
- calculating RG flow hard at $g_i \approx 1$

Alternative: conformal bootstrap

- spectrum of local ops $\mathcal{O}_i(x)$: $(\Delta_i, P_i)_{i=0}^{\infty}$
- structure constants $\mathcal{O}_i \times \mathcal{O}_j = \sum_k c_{ijk} \mathcal{O}_k$
- impose associativity $(\mathcal{O}_i \times \mathcal{O}_j) \times \mathcal{O}_k = \mathcal{O}_i \times (\mathcal{O}_j \times \mathcal{O}_k)$

$\Rightarrow$ Precise mathematical definition of CFT

$$\text{RG CFT} \overset{?}{\doteq} \text{Bootstrap CFT}$$

$\Rightarrow$ uncontroversial

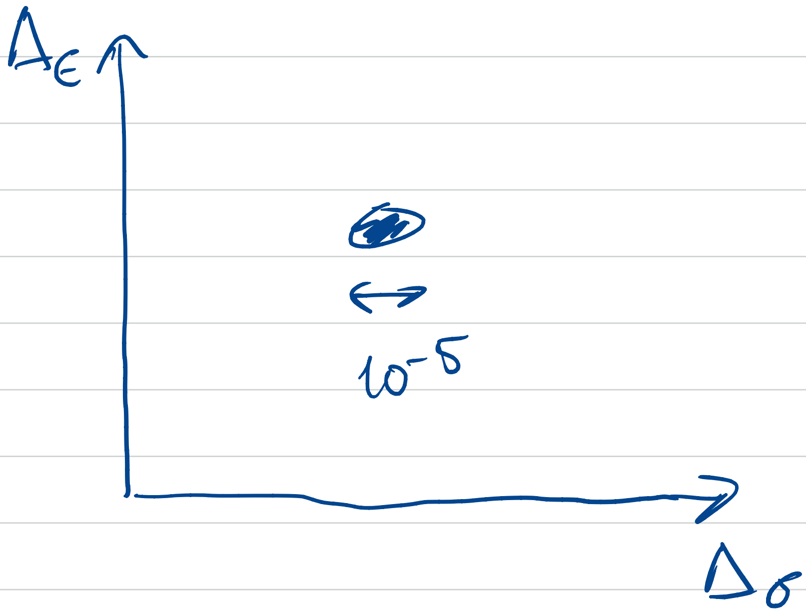
$\Leftarrow$ much less clear

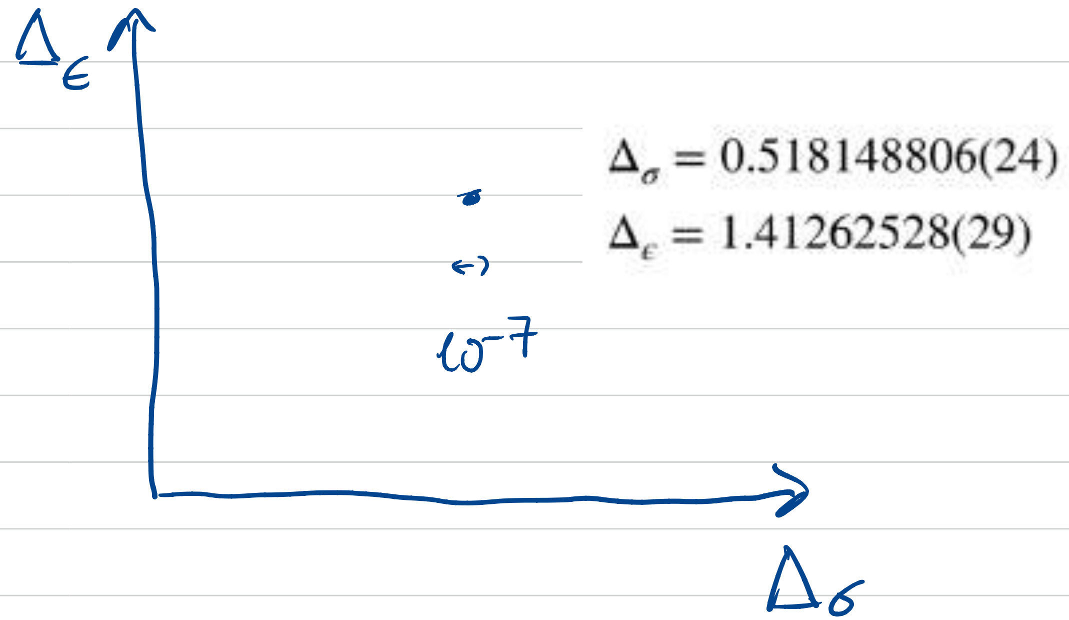
Are all solutions of the bootstrap physical?

In other words, is the conformal bootstrap complete?

evidence: 3d Ising bootstrap

El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi 2012, 2014





$\{\sigma, \epsilon\}$

Kos, Poland, Simmons-Duffin
2014, 2016

$\{\sigma, \epsilon, T_{\mu\nu}\}$

Chang, Dommes, Erramilli,
Homrich, Kravchuk, Liu,
Mitchell, Poland, Simmons-Duffin
2024

If the bootstrap is indeed complete:

- $d=4$, $\mathcal{N}=4$ $\Rightarrow$ SYM (?)

  Beem, Rastelli, van Rees 2013, 2016
  Chester, Dempsey, Pufu 2021

- $d=6$, $\mathcal{N}=(2,0)$ $\Rightarrow$ classified by ADE ?

  Beem, Lemos, Rastelli, van Rees 2015

- $d>6$ $\Rightarrow$ free theory ??

- $T_{\mu\nu}$, no $\Delta=d$ scalar op. $\Rightarrow$ isolated solution

Today:

2 pieces of evidence for completeness of the bootstrap

1. 1d long-range Ising model

Benedetti, Lauria, DM, van Vliet: 2024 + WIP

2. Spectral theory of hyperbolic manifolds

Kravchuk, DM, Pal: 2021

Bonifacio, DM, Pal: 2023

Adve: to appear

① 1d long-range Ising model

Benedetti, Lauria, DM, van Vliet : 2024 + WIP

- classical statistical model on $\mathbb{Z}$ lattice
- $\sigma : \mathbb{Z} \to \{-1, 1\}$

$$H_s(\{\sigma_i\}) = \frac{1}{8} \sum_{i<j} \frac{(\sigma_i - \sigma_j)^2}{|i-j|^{1+s}}$$

- If $0 < s \leq 1$, there is $0 < T_* < \infty$ s.t.

$T < T_* \Rightarrow \langle \sigma_i \rangle \neq 0$ Dyson 1969

$T > T_* \Rightarrow \langle \sigma_i \rangle = 0$ Thouless 1969

- $T = T_*$ is a continuous phase transition described by a 1d CFT

Paulos, Rychkov, van Rees, Zan 2015

- investigate this CFT!

symmetries:

- conformal $PSL_2(\mathbb{R})$ $\quad x \mapsto \dfrac{ax+b}{cx+d}$
- global $\mathbb{Z}_2$: $\quad \sigma(x) \to -\sigma(x)$
- parity: $\quad \sigma(x) \mapsto \sigma(-x)$

local primary operators $\quad \mathcal{O}_i(x) \quad i = 0, 1, 2, \ldots$

Task: study the CFT data $\Delta_i(s)$, $c_{ijk}(s)$ for $0 < s < 1$.

The simplest CFT not (yet) solved analytically?

A continuum UV description:

$$S = \# \iint \frac{[\varphi(x)-\varphi(y)]^2}{|x-y|^{1+s}} \, dx\, dy + \int \left(\# \varphi^2(x) + \# \varphi^4(x) \right) dx$$

$$\underbrace{\phantom{\iint \frac{[\varphi(x)-\varphi(y)]^2}{|x-y|^{1+s}} dx\,dy}}$$

not renormalized $\Rightarrow$ $\Delta_\varphi = \Delta_\sigma = \dfrac{1-s}{2}$ protected

$0 < s < \frac{1}{2} \Rightarrow \varphi^4$ irrelevant $\Rightarrow \text{Ising}_s = \text{GFF}_{\frac{1-s}{2}}$

$\frac{1}{2} < s < 1 \Rightarrow \varphi^4$ relevant $\Rightarrow \text{Ising}_s =$ interacting 1d CFT

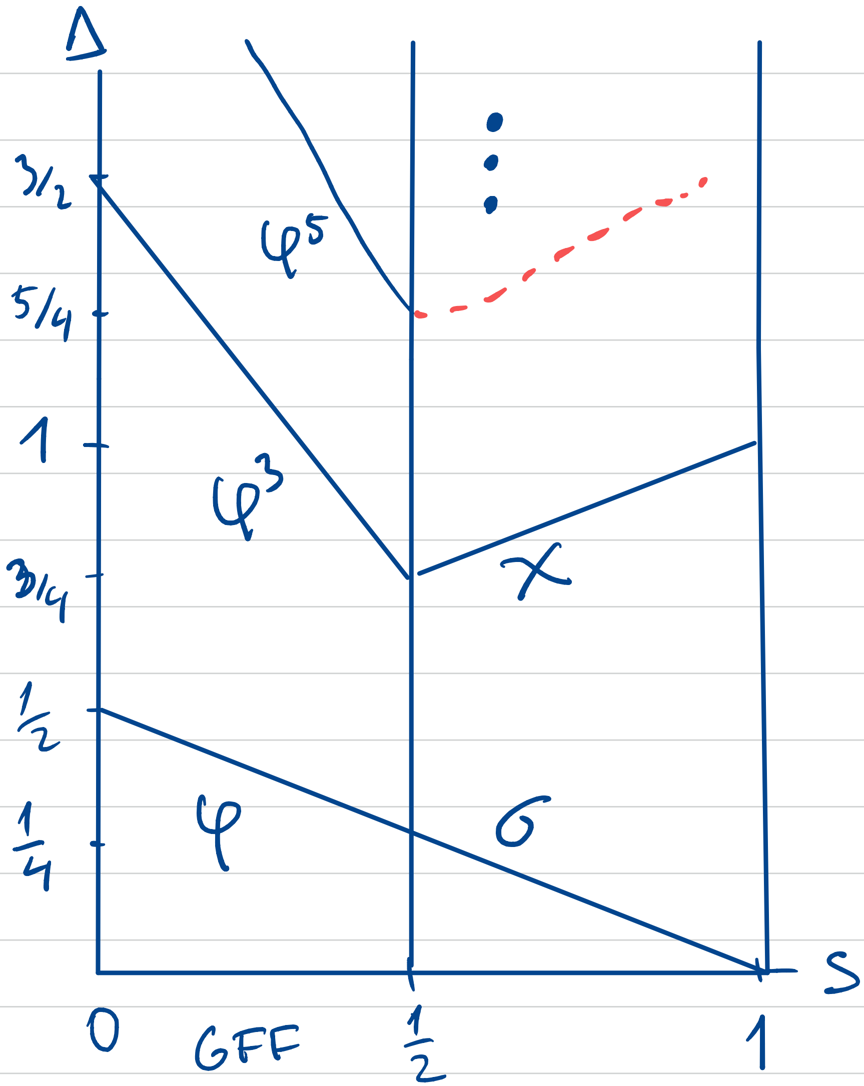
Equations of motion:

$$\partial_x^s \varphi \sim \lambda \varphi^3 =: \chi$$

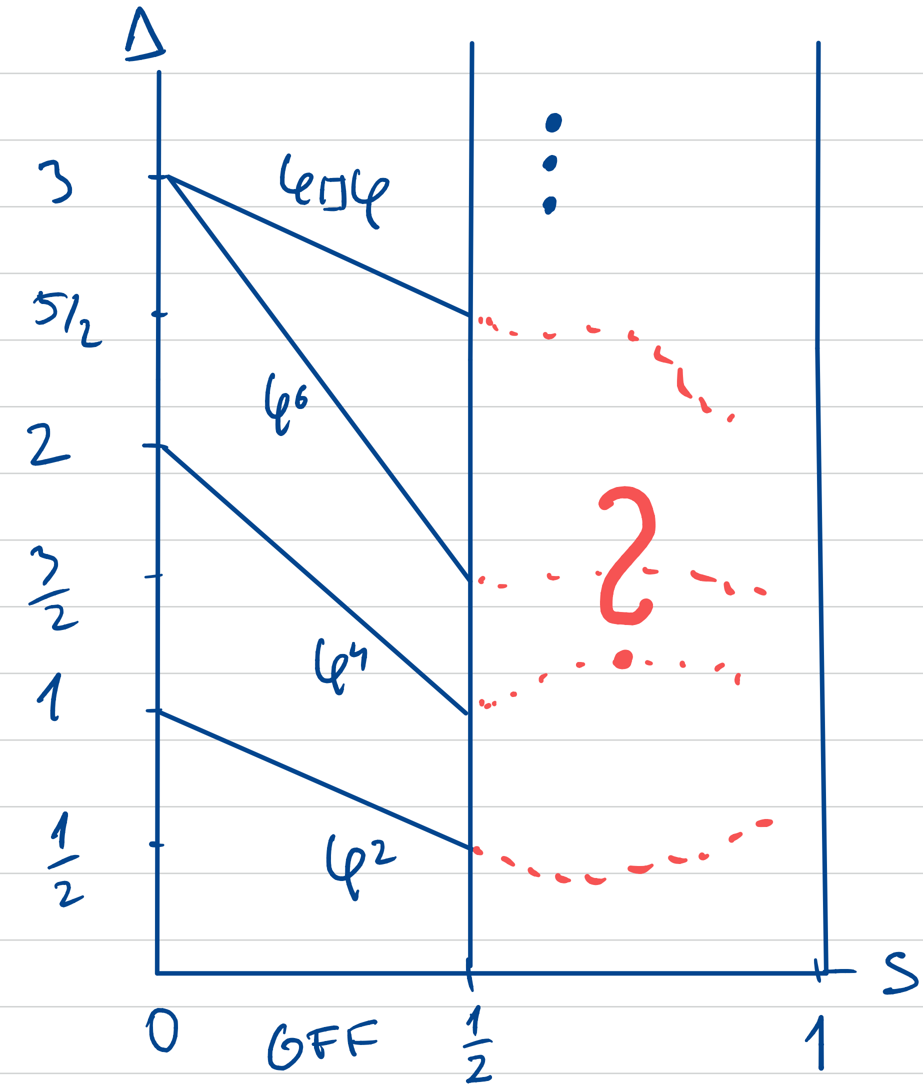
primary

$\Delta_\chi = \Delta_\sigma + s = \dfrac{1+s}{2}$ protected

$$\boxed{\Delta_\sigma + \Delta_\chi = 1}$$



$\mathbb{Z}_2$-odd, parity-even



$\mathbb{Z}_2$-even, parity-even

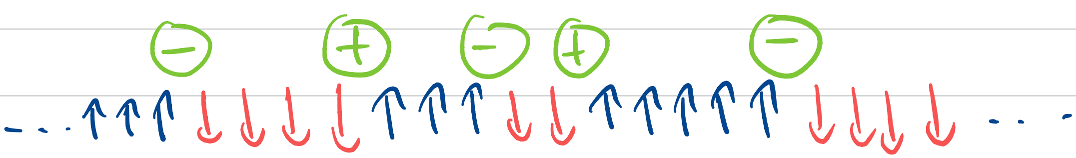
Our work: Exact solution of Ising at $s=1$

+ systematic perturbation theory in $\delta = 1-s$

- physics near $s=1$: domain walls are dilute
- rewrite H_s in terms of $q_i = \frac{\sigma_i - \sigma_{i-1}}{2} = \begin{cases} 0 & \uparrow\uparrow \\ 1 & \downarrow\uparrow \\ -1 & \uparrow\downarrow \end{cases}$

$$H_s \approx -\sum_{i<j} \log|i-j| \, q_i q_j$$

← 1d Coulomb gas



opposite charges attract
like charges repel

- $\Delta_\sigma = \frac{1-s}{2} \to 0$ as $s \to 1$

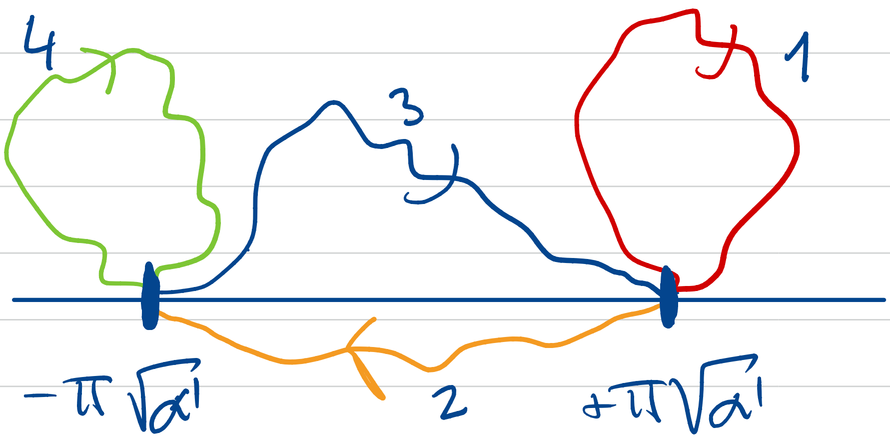
$\Rightarrow$ σ = topological op. ~ 1d short-range Ising

$\sigma|\pm\rangle = \pm|\pm\rangle$
$\sigma^2 = \mathbb{1}$

Our proposal

Benedetti, Lauria, DM, van Vliet 2024

$$\boxed{\begin{array}{l}\text{critical 1d long range Ising @ } s=1 \\ \qquad\qquad = \\ \text{conformal boundary condition for} \\ \text{2d noncompact free scalar } \phi: \\ \text{2 D0-branes at distance } 2\pi\sqrt{\alpha'}\end{array}}$$



$$\begin{pmatrix} \mathcal{O}_1 & e^{i\phi}\mathcal{O}_2 \\ e^{-i\phi}\mathcal{O}_3 & \mathcal{O}_4 \end{pmatrix}$$

stringy interpretation due to J. Vošmera

$$\mathcal{O}_j = (\partial\phi)^{n_1} \dots (\partial^N \phi)^{n_N}$$

$\mathbb{Z}_2$- symmetry: $\quad \phi \to -\phi \qquad \hat{A} \mapsto \hat{\sigma}_1 \hat{A} \hat{\sigma}_1$

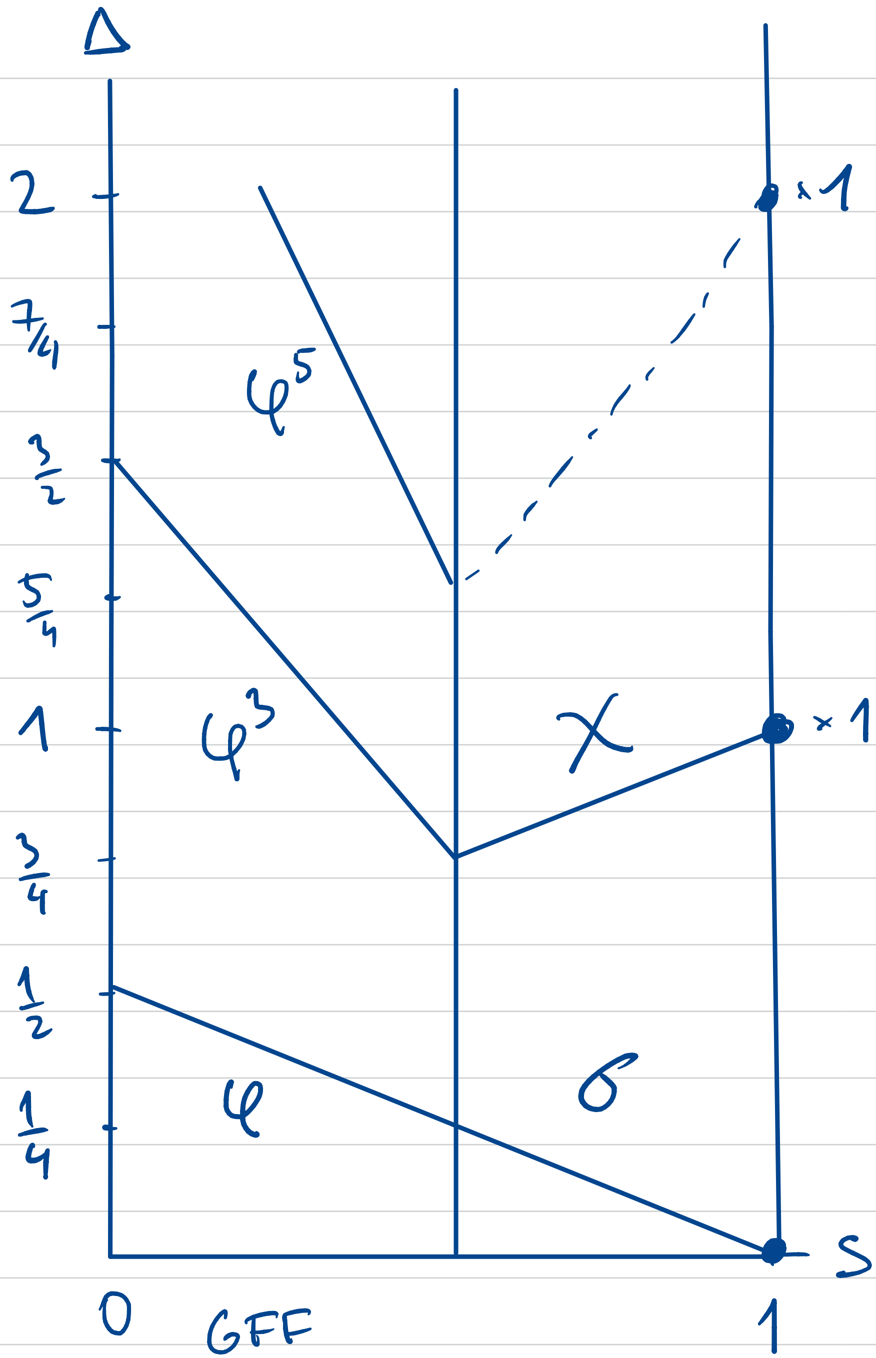
Low-lying operators:

$\Delta = 0$: $\quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

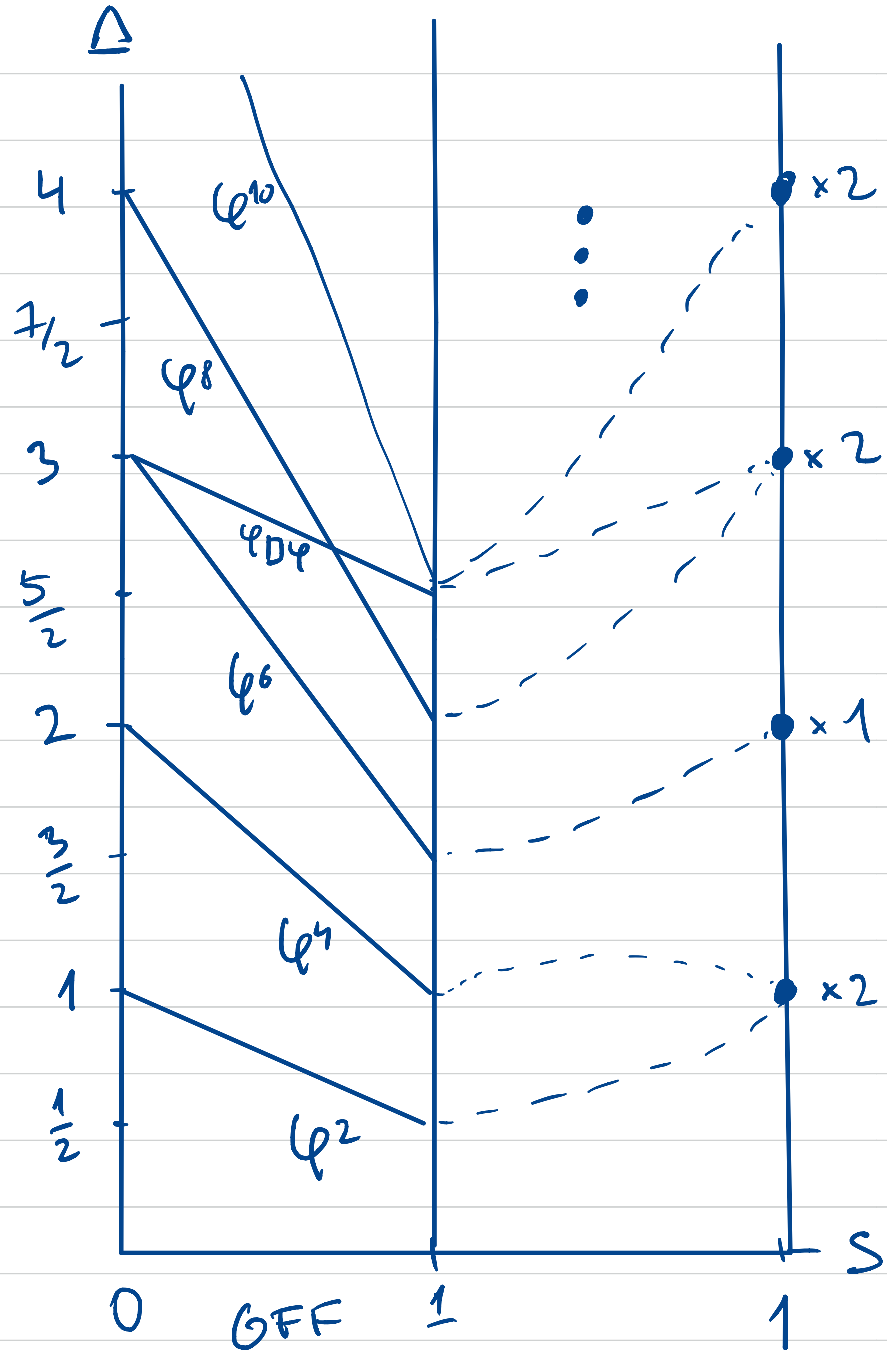
$\Delta = 1$: $\quad \chi = \begin{pmatrix} \partial\phi & 0 \\ 0 & \partial\phi \end{pmatrix}, \qquad \mathcal{O}_\pm = \begin{pmatrix} \pm\partial\phi & e^{i\phi} \\ e^{-i\phi} & \mp\partial\phi \end{pmatrix}$

$$\sigma' = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}$$

$\Rightarrow$ complex solution of the 1d Ising CFT at $s=1$.



$\mathbb{Z}_2$ − , parity +



$\mathbb{Z}_2$ + , parity +

- what about $s < 1$? $\to$ field theory

  $\to$ analytic conformal bootstrap

- bootstrap assumptions

1) Δ_i, C_{ijk} asymptotic series in $\sqrt{1-s}$

2) $s=1$ reproduces the exact solution

3) $\Delta_\sigma = \frac{1-s}{2}$, $\Delta_\chi = \frac{1+s}{2}$

4) crossing equations for:

$$\langle \sigma\sigma\sigma\sigma \rangle \qquad \langle \sigma\sigma \mathcal{O}_\pm \mathcal{O}_\pm \rangle$$
$$\langle \sigma\sigma\sigma\chi \rangle \qquad \langle \sigma \mathcal{O}_\pm \sigma \mathcal{O}_\pm \rangle$$
$$\langle \sigma\sigma\chi\chi \rangle \qquad \langle \sigma\chi \mathcal{O}_\pm \mathcal{O}_\pm \rangle$$
$$\langle \sigma\chi\sigma\chi \rangle \qquad \langle \sigma \mathcal{O}_\pm \chi \mathcal{O}_\pm \rangle$$
$$\langle \sigma\chi\chi\chi \rangle$$

Results :

$\delta = 1 - S$

$$\Delta_\pm = 1 \pm \sqrt{2\delta} + \delta/4 + O(\delta^{3/2})$$

(agrees with field theory)

$$c_{\sigma\sigma\pm} = \pm\frac{\sqrt{\delta}}{\sqrt{2}} - \frac{15}{16}\delta \mp \left(\frac{\pi^2}{3} - \frac{543}{128}\right)\frac{\delta^{3/2}}{2^{3/2}} + O(\delta^2)$$

$$c_{\sigma\chi\pm} = \frac{1}{\sqrt{2}} \mp \frac{\sqrt{\delta}}{16} - \frac{\delta}{256\sqrt{2}} + O(\delta^{3/2})$$

$$c_{\chi\chi\pm} = -\frac{\pi^2\delta}{2} \mp \frac{11\pi^2\delta^{3/2}}{16\sqrt{2}} + O(\delta^2)\,.$$

$$c_{---} = +\frac{3}{2} - \frac{39}{16\sqrt{2}}\sqrt{\delta} + O(\delta)$$

$$c_{--+} = -\frac{1}{2} - \frac{1}{16\sqrt{2}}\sqrt{\delta} + O(\delta)$$

$$c_{-++} = -\frac{1}{2} + \frac{1}{16\sqrt{2}}\sqrt{\delta} + O(\delta)$$

$$c_{+++} = +\frac{3}{2} + \frac{39}{16\sqrt{2}}\sqrt{\delta} + O(\delta)$$

- Uniquely fixed by the bootstrap to this order.

- OPE relations $\dfrac{c_{\sigma ab}\, c_{\chi cd}}{c_{\chi ab}\, c_{\sigma cd}}$ automatic !

Paulos, Rychkov, van Rees, Zan 2015

(2) Spectral theory of hyperbolic manifolds

Kravchuk, DM, Pal : 2021

Bonifacio, DM, Pal : 2023

Adve : to appear

- Hyperbolic manifold = Riemannian manifold with curvature $= -1$
- Hyperbolic space $\mathbb{H}^D = SO^+(D,1)/SO(D)$
- General case $M = \Gamma \backslash \mathbb{H}^D$, $\Gamma < SO^+(D,1)$
- Laplace eq$^{\text{n}}$ $-\nabla^2 h_i(x) = \lambda_i h_i(x)$
- spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

spectral gap

quantum chaos

Deep analogy between

spectral theory of hyperbolic $(d+1)$-manifolds	conformal field theory in d dimensions

$$\boxed{SO(d+1,1)}$$

isometries of $\mathbb{H}^{d+1}$	conformal automorphisms of S^d
Laplace eigenfunction h_i	primary operator $\mathcal{O}_i$
eigenvalue λ_i	Casimir $\lambda_i = \Delta_i(d-\Delta_i)$
rank J_i of tensor bundle	spin J_i
$c_{ijk} = \int_M h_i h_j h_k \, dx^{d+1}$	OPE coefficients c_{ijk}
$L^2(\Gamma \backslash SO(d+1,1))$	Hilbert space

Main idea: Formulate bootstrap constraints on spectra of hyp. mfds using rep^{n} theory of $SO(d+1,1)$.

- Given $M = \Gamma \backslash \mathbb{H}^{d+1}$, consider $L^2(\Gamma \backslash SO(d+1,1))$

$$L^2(\Gamma \backslash SO(d+1,1)) = \bigoplus_i R_{\Delta_i, J_i}$$

∞-dim^{l} irreps of

- $(\Delta_i, J_i) \Leftrightarrow$ Laplace spectrum

- Multiplication map $C^\infty(\Gamma \backslash G) \times C^\infty(\Gamma \backslash G) \to C^\infty(\Gamma \backslash G)$ defines an OPE

- associativity $\Rightarrow$ constraints on the spectrum

Results for hyperbolic 2-manifolds

<u>Theorem</u> (Bonifacio, Kravchuk, DM, Pal) :

genus = 2 $\Rightarrow$ $\lambda_1 \leq 3.83890$

genus = 3 $\Rightarrow$ $\lambda_1 \leq 2.679$

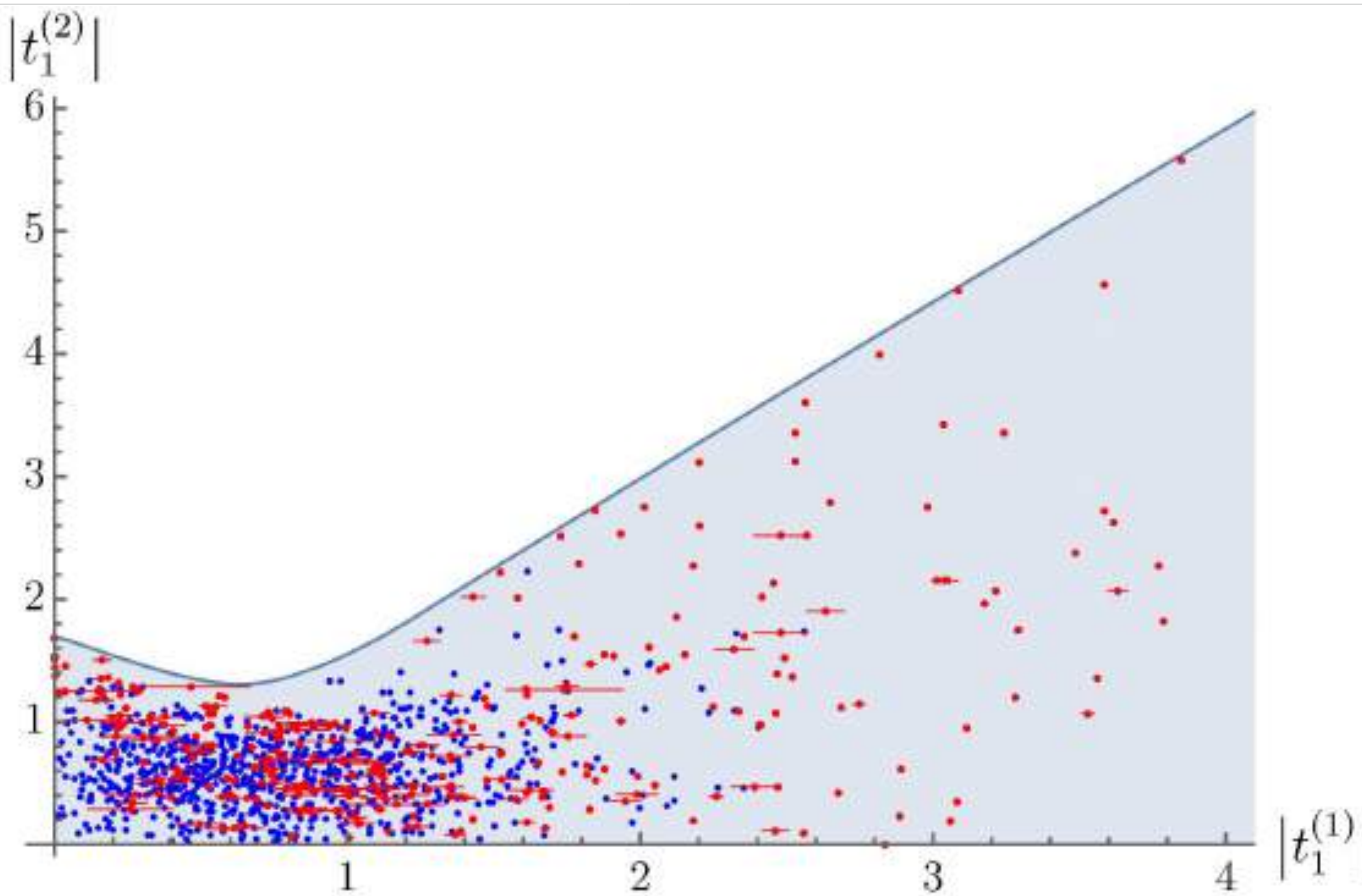
- There exist $\rightarrow$ genus = 2 surface with $\lambda_1 \approx 3.83889$

  $\rightarrow$ genus = 3 surface with $\lambda_1 \approx 2.678$

- A. Radcliffe (2024): Evidence that the optimal bootstrap bound from single 4-pt function is not exactly saturated.

Results for spectra of hyperbolic 3-manifolds

spectral gap of ∇^2 on rank-2 tensor fields



spectral gap of ∇^2 on vector fields

- Why is the bootstrap method so powerful?
- Is the bootstrap of hyperbolic manifolds <u>complete</u>?

Yes!

<u>Theorem</u> (A. Adve 2025):

"Every solution of the hyperbolic bootstrap equations arises from a hyperbolic manifold."

[So far for compact hyp. 2-manifolds.]

Sketch of the proof:

① The hyperbolic bootstrap equations axiomatize

$\Rightarrow$ V = unitary rep. of $SO(2,1)$

$\Rightarrow$ V carries a commutative, associative product

② Gelfand duality:

abelian algebra = functions on a space

③ In the case ①, Gelfand duality produces

$$V = L^2(\Gamma \backslash SO(2,1)).$$

- It follows we can define a hyperbolic manifold either in the usual way, or as a solution of the hyperbolic bootstrap.
- While the latter is useful, it obscures the essence of what a hyp. manifold is.
- For CFTs, we currently only have the bootstrap definition.
- What is the CFT analogue of the geometric definition?

object	algebraic definition	geometric definition
hyperbolic manifold	a solution of the hyperbolic bootstrap	$\Gamma \backslash \mathbb{H}^D$
conformal field theory	a solution of the conformal bootstrap	?